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ALGEBRA

Equations, Polynomials

Ostrowski, Alexandre. Sur la continuité relative des racines d'équations algébriques. C. R. Acad. Sci. Paris **209**, 777–779 (1939). [MF 862]

Comparing the roots $1.0872\cdots$, $.9198\cdots$, $.9965\cdots \pm i\cdot.0836\cdots$ of $z^4-4z^3+5.999951z^2-4z+1=0$ with the roots 1, 1, 1, 1 of $z^4-4z^3+6z^2-4z+1=0$, we see that a change of 1/1000 of 1% in one of the coefficients effects a change of about 8% in the roots. To estimate the influence of variation of the coefficients on the roots, two theorems are given, with indication of the method of proof.

I. Let x_ν, y_ν represent, in appropriate order, the roots of $z^n+a_1z^{n-1}+\cdots+a_n=0$ and $z^n+b_1z^{n-1}+\cdots+b_n=0$, respectively, and let $T=\max(1, |a_\nu|^{1/\nu}, |b_\nu|^{1/\nu})$, $\nu=1, \cdots, n$, and let $\delta=(|a_1-b_1|^2+\cdots+|a_n-b_n|^2)^{1/2}$. Then $|y_\nu-x_\nu| \leqq 4nT\delta^{1/n}$, $\nu=1, \cdots, n$.

II. The second theorem is copied from the paper: Soient x_ν et y_ν respectivement les racines des polynomes $f(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n$, $g(z)=b_0z^n+b_1z^{n-1}+\cdots+b_n$, où $a_0a_nb_0b_n \neq 0$, et supposons que l'on ait $b_\nu-a_\nu=\theta_\nu a_\nu$, $|\theta_\nu|=\tau$ ($\nu=0, 1, \cdots, n$), où $16n\tau^{1/n} \leqq 1$. Alors on a, en numérotant les x_ν et y_ν convenablement $|1-(y_\nu/x_\nu)| \leqq 15n\tau^{1/n}$ ($\nu=1, \cdots, n$).

A. J. Kempner (Boulder, Colo.).

Colucci, Antonio. Sopra i polinomi definiti e le equazioni algebriche a coefficienti complessi. Rend. Sem. Mat. Roma **3**, 84–95 (1939). [MF 503]

If two polynomials $u(z)$, $v(z)$ of nth degree have real coefficients and if all of their zeros are real and separate each other, then the imaginary parts of all the zeros of the polynomial $u(z)+iv(z)$ have the same sign, and conversely. This and related theorems are proved here in an elementary way. The simple proof given for a theorem of A. Hurwitz coincides with one given by the reviewer in the Math. Z. **11**, 27–28 (1921). *O. Szász* (Cincinnati, Ohio).

Romanini, Clelia. Sulla realità delle radici di un'equazione di quarto grado. Period. Mat. **19**, 198–220 (1939). [MF 916]

Cantoni, Riccardo. L'equazione dei pentagoni articolati inscrittibili. Parte II. Il discriminante dell'equazione simmetrica. Ist. Lombardo, Rend. **72**, 165–178 (1939). [MF 949]

Questo lavoro è la continuazione di quello presentato nell'adunanza del 20 maggio 1937 ed in esso si studia il discriminante dell'equazione dei pentagoni articolati del quale si calcolano i fattori triviali mentre per l'unico fattore essenziale si dà soltanto il significato e la dimostrazione d'irriducibilità. *Summary.*

Tietze, Heinrich. Über symmetrische Funktionen von endlich oder abzählbar unendlich vielen Veränderlichen. Monatsh. Math. Phys. **48**, 487–499 (1939). [MF 663]

This is an exposition of the fundamental theorem on symmetric functions after the manner of proof given by Perron [Algebra, 1927, p. 152]. The present communication is preliminary to a further installment.

H. W. Turnbull (St. Andrews).

Palamà, Giuseppe. Sulla risolvente razionale della $\sum_{i=1}^{n}\sqrt{a_i}=0$, sotto forma di determinante. Period. Mat. **19**, 265–270 (1939). [MF 902]

To find the rationalization indicated in the title, the author constructs a determinant of order 2^{n-1}, thus generalizing a device of Cayley. *O. Szász* (Cincinnati, Ohio).

Skolem, Th. Some remarks on decomposition of polynomials into sums of two squares. Norsk Mat. Tidsskr. **21**, 154–159 (1939). (Norwegian) [MF 870]

Linear Algebra

Bennett, G. T. Continuants and precontinuants. Proc. Cambridge Philos. Soc. **35**, 548–561 (1939). [MF 835]

In this paper the continuant $u_{n+1}=K(a_1, a_2, \cdots, a_n)$, defined by the difference equations $u_{n+1}=a_nu_n+u_{n-1}$ together with the initial conditions $u_0=0$, $u_1=1$, is denoted by $(1, n)$. The new series of continuants, obtained from the same difference equations by replacing the initial conditions by $u_m=0$, $u_{m+1}=1$, is taken as the $(m+1)$st row of a matrix, whose elements are continuants. This matrix becomes skew symmetric if each odd numbered row is multiplied by -1. The element in the mth row and nth column of this skew symmetric matrix is $[m, n]$, where $[n, n]=0$, $[n, n+1]=(-1)^n$, $[n, m]=-[m, n]$ and $[m, n]=(-1)^m(m+1, n-1)$. By means of these symbols and the vanishing of subdeterminants of the matrix many identities are established, including $[m, n][l, r]+[n, l][m, r]+[l, m][n, r]=0$, a symmetric form of Euler's continuant identity. The special case, in which all parameters a_i have the same value a, is then considered and, in the last section, it is shown how continuants, given by the more general chain of equations $u_{n+1}=a_nu_n+b_nu_{n-1}$, may be treated in the same manner.

J. Williamson (Baltimore, Md.).

Potron, l'Abbé. Sur les matrices non négatives, et les solutions positives de certains systèmes linéaires. Bull. Soc. Math. France **67**, 56–61 (1939). [MF 445]

This paper is divided in two parts. The first part gives a lucid exposition of the recursion method of C. Lanczos for solving a system of n linear equations in n unknowns. Lanczos has so far published only a brief abstract on the subject [Bull. Amer. Math. Soc. **42**, 325 (1936)] but has sent his detailed manuscript to the author. In the second part the method of Lanczos is applied to a system of the type $\sum_k(s\delta_{ik}-a_{ik})x_k=b_i$, $(i, k=1, 2, \cdots, n)$, where $a_{ik} \geqq 0$, $b_i \geqq 0$, $s>0$. It is shown that the results of the numerical work involved in the solution allow the reader to decide

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(early, in general) whether s is less than or greater than the maximal positive root r of the equation $\det|r\delta_{ik}-a_{ik}|=0$. Economic problems concerning the equilibrium of production and consumption or price and salaries have led the author to systems of this type with $s=1$, a condition on the a_{ik} being that r be less than 1. Thus Lanczos' method does not only conveniently solve the systems involved, but also allows the reader to recognize early in the computations whether the condition $r<1$ is satisfied or not.

I. J. Schoenberg (Waterville, Me.).

Rella, Tonio. Über positiv-homogene Funktionen ersten Grades einer Matrix. Monatsh. Math. Phys. **48**, 84–95 (1939). [MF 628]

The author studies some generalizations of the notion of absolute value of a square matrix with complex elements. Let $\varphi(\mathfrak{A})$ be a function having real nonnegative values, defined for each matrix $\mathfrak{A}$ and satisfying the following postulates: (I) $\varphi(\gamma\mathfrak{A})=|\gamma|\varphi(\mathfrak{A})$, for complex γ; (II) φ is continuous; (IIIa) $\varphi(\mathfrak{A})=0$ if and only if $\mathfrak{A}$ is nilpotent; (IVa) $\varphi(\mathfrak{A}^2)=\varphi^2(\mathfrak{A})$. Then it is shown that $\varphi(\mathfrak{A})=|\lambda_1|$, where λ_1 is a characteristic root of $\mathfrak{A}$ of greatest absolute value, and this uniquely defined function is denoted by $|\mathfrak{A}|$. If postulates IIIa and IVa are respectively replaced by (IIIb) $\varphi(\mathfrak{A})=0$ if and only if $\mathfrak{A}=0$, and (IVb) $\varphi^2(\mathfrak{A})=\varphi(\bar{\mathfrak{A}}\mathfrak{A})$, where $\bar{\mathfrak{A}}$ is the hermitian conjugate of $\mathfrak{A}$; then it is shown that $\varphi(\mathfrak{A})=(|\bar{\mathfrak{A}}\mathfrak{A}|)^{1/2}$. Both sets of postulates are independent. Some other related functions are also considered. For example, if $a_1^2\geqq a_2^2\geqq\cdots\geqq a_n^2\geqq 0$ are the characteristic roots of $\bar{\mathfrak{A}}\mathfrak{A}$, and k is a fixed positive integer not exceeding n, the function $\varphi_k(\mathfrak{A})=(a_1^2+a_2^2+\cdots+a_k^2)^{1/2}$ satisfies postulates I, II and IIIb. Further, $\varphi_k(\mathfrak{A}+\mathfrak{B})\leqq\varphi_k(\mathfrak{A})+\varphi_k(\mathfrak{B})$ and $\varphi_k(\mathfrak{A}\mathfrak{B})\leqq\varphi_k(\mathfrak{A})\varphi_k(\mathfrak{B})$. In particular, $\varphi_n(\mathfrak{A})$ is the usual absolute value of $\mathfrak{A}$ [cf. J. H. M. Wedderburn, Lectures on Matrices, New York, 1934, p. 125]. The author obtains some inequalities between these various functions and concludes with some applications, principally to convergence of series.

N. H. McCoy.

Rados, Gusztáv. Über semidefinite Hermite-Formen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **58**, 288–297 (1939). (Hungarian. German summary) [MF 1017]

Two proofs are given for the theorem that the determinant of a semi-definite Hermitian form vanishes.

O. Szász (Cincinnati, Ohio).

Williamson, John. Note on a principal axis transformation for non-hermitian matrices. Bull. Amer. Math. Soc. **45**, 920–922 (1939). [MF 783]

Let A and B be two $r\times s$ matrices with complex elements. Necessary and sufficient conditions that there exist unitary matrices U and V such that UAV and UBV are diagonal are that

$$AB^*=f(BA^*), \qquad B^*A=f(A^*B),$$

where f is a polynomial and A^* is A conjugate transposed.

C. C. MacDuffee (Madison, Wis.).

Williamson, John. The exponential representation of automorphs of a symmetric or hermitian matrix. Amer. J. Math. **62**, 153–164 (1940). [MF 965]

First, let H be a nonsingular hermitian matrix. If G is anti-hermitian, it is known that $C=\exp HG$ is a conjunctive automorph of H, that is, that $CHC^*=H$, where C^* is the conjugate transposed of C. It is now proved, conversely, that every conjunctive automorph of H is of this form. Secondly, let H be nonsingular and symmetric over the complex field, and let C^* be the transposed of the complex matrix C. If $CHC^*=H$, then $C=\exp HG$ for some skew symmetric matrix G if and only if no elementary divisor $(\lambda+1)^{2k+1}$ occurs an odd number of times among the elementary divisors of C. If $|C|=1$, $C=\exp HG$ or else is the limit of matrices of this form. For every automorph C of H which is not of exponential form, there exists an automorph D of period 2 such that DC is exponential. Thirdly, the case where H is symmetric over the real field is completely discussed, but the results are more involved.

C. C. MacDuffee (Madison, Wis.).

Radon, Johann. Ein Satz der Matrizenrechnung und seine Bedeutung für die Analysis. Monatsh. Math. Phys. **48**, 198–204 (1939). [MF 640]

Let A and B be square matrices of order n, and (A, B) the matrix of n rows and $2n$ columns obtained by juxtaposition of A and B. The author proves that, if A and B are given, there always exists a symmetric matrix S such that the rank of $AS+B$ is the same as the rank of (A, B). The proof makes use of continuity and applies to matrices over the real or complex field. The special case in which the ranks are n is then used to prove the known result that, if the alternating bilinear form $x_iy_{n+i}-y_ix_{n+i}$ $(i=1, 2, \cdots, n)$ is transformed into itself by the substitutions $x_i=u_{ik}X_k$, $y_i=u_{ik}Y_k$ $(i, k=1, 2, \cdots, 2n)$, then $|u_{ik}|=1$ [cf. Carathéodory, Variationsrechnung und partielle Differentialgleichungen, 1935, p. 90]. If A and B are given matrices such that the rank of (A, B) is n, and $AB'=BA'$, then there always exist matrices C and D such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

transforms the above bilinear form into itself. Following a previous paper [Abh. Math. Sem. Hans. Univ. **6**, 273–299 (1928)], the author applies this result to the problem of Lagrange without side conditions, and obtains a theorem concerning the multiplicity of a focal point of a conjugate system [cf. Morse, The Calculus of Variations in the Large, 1934, p. 47].

N. H. McCoy (Northampton, Mass.).

Tognetti, Mario. Sulle omografie permutabili con una omografia singolare. Ist. Lombardo, Rend. **72**, 309–322 (1939). [MF 928]

By reducing the matrix A of a singular collineation to canonical form, necessary and sufficient geometric conditions that a nonsingular collineation B be permutable with A are determined. In the simplest case, where A is of order n and is nilpotent of order n, these conditions are that B transform a group of characteristic points of A (that is, a group of points A^ix, $i=0, 1, 2, \cdots, n-1$) into another such group. Similar conditions are obtained in the more general case.

J. Williamson (Baltimore, Md.).

Rados, Gusztáv. Über cyklische unitäre Substitutionen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **58**, 277–287 (1939). (Hungarian. German summary) [MF 1016]

The main result is: A necessary and sufficient condition that a unitary substitution A be cyclic is that there exists a unitary substitution P such that all coefficients of the transformed substitution $P^{-1}AP$ are algebraic integers.

O. Szász (Cincinnati, Ohio).

Oldenburger, Rufus. Factorability of general symmetric matrices. Compositio Math. 7, 223–228 (1939). [MF 796]

A symmetric p-way matrix $A=(a_{ij\cdots m})$ is called factorable if numbers λ_i and a square matrix $B=(b_{ij})$ exist such that

$$A=(\sum_{\alpha=1}^{\sigma}\lambda_\alpha b_{\alpha i}b_{\alpha j}\cdots b_{\alpha m}).$$

The author states that A is factorable in a field K if and only if K contains at least p distinct elements. Details of proof are omitted. *C. C. MacDuffee* (Madison, Wis.).

Rinehart, R. F. An interpretation of the index of inertia of the discriminant matrices of a linear associative algebra. Trans. Amer. Math. Soc. 46, 307–327 (1939). [MF 466]

The discriminant matrix of a linear algebra $\mathfrak{A}$ over the real number field has two invariants, rank and signature. This paper completes the interpretation of these invariants in terms of the algebra. For any element a in $\mathfrak{A}$, the first trace $t_1(a)$ is the trace of the matrix representing a in the first regular representation. If $\{b\}=\{b_1, \cdots, b_n\}$ is a basis for $\mathfrak{A}$, then the matrix of traces $\|t_1(b_rb_s)\|=T_1(\mathfrak{A})$ is known as the first discriminant matrix of $\mathfrak{A}$ relative to the given bases [C. C. MacDuffee, Ann. of Math. (2) 32, 60–66 (1931); E. Noether, Math. Z. 30, 689 (1929)]. A change of the basis alters the discriminant matrix to a congruent matrix. Hence the rank ρ and signature σ of this matrix $T_1(\mathfrak{A})$ are invariants of the algebra. One may also use as the invariants the nullity $n-\rho$ and the "index of inertia" $\mu=n+(\sigma-\rho)/2$, defined as the number of nonnegative terms in a diagonal form of the matrix. It is known that the nullity is the order of the radical of $\mathfrak{A}$ [Dickson, Algebra, 108]. The author now proves that the index of inertia μ is the least order of all those subalgebras of $\mathfrak{A}$ which (i) contain a complete set of primitive idempotents; (ii) have as radical a maximal nilpotent subalgebra of $\mathfrak{A}$. Alternatively, $\mu=\epsilon+\chi$, where ϵ is the number of primitive idempotents in a complete set, χ the order of a maximal nilpotent subalgebra of $\mathfrak{A}$.

The proof depends on the Wedderburn structure theorems, computing at each stage of the analysis of $\mathfrak{A}$ the consequent formulas for μ, ϵ and χ. The chief difficulty arises in the computation of μ for an algebra $\mathfrak{A}=\mathfrak{A}^*+\mathfrak{R}$ with radical $\mathfrak{R}$, given the value of μ for the semi-simple algebra $\mathfrak{A}^*$. This difficulty the author meets by choosing for $\mathfrak{A}$ an elaborate canonical basis. The conclusion $\mu=\epsilon+\chi$ contains as a special case part of the Borchardt-Jacobi Theorem, which expresses the number of real roots of a polynomial $f(x)$ as the signature of a certain matrix [cf. Rinehart, Bull. Amer. Math. Soc. 42, 570–576 (1936)]. *S. MacLane.*

Abstract Algebra

***Albert, A. Adrian. Structure of Algebras.** American Mathematical Society Colloquium Publications, vol. 24. American Mathematical Society, New York, 1939. xi+210 pp. $4.00.

In recent years the theory of algebras and hypercomplex numbers has been an active and fertile diversion of modern mathematics. The new book by Professor Albert is therefore a very timely and valuable document. The book gives an extensive account of the present state of the theory including the most recent development. It appears that the whole domain has reached a more mature and clarified form through this work. Anybody familiar with the subject will detect considerable improvement even in some of those parts of the theory which one now considers classical. The value of the book is further enhanced by a complete biography. The reviewer has only one observation to make which applies not only to this book, but to several others which have recently appeared. It seems preferable, wherever possible, to make the historical development an integral part of the presentation. It adds to the flavor of the book and makes it easier for an unfamiliar reader to decide to whom the various results should be ascribed.

The first chapter gives the more fundamental concepts and lemmas. It is assumed everywhere that the algebras in question have finite bases. One finds the simplest properties of linear sets and transformations, matrix representations, and finally an important lemma by Wedderburn. The second chapter gives the theory of ideals and their principal decomposition theorems. The radical is examined and the existence of idempotents is established preparatory to the proof of Wedderburn's representation theorem. This follows in the next chapter, together with some of the most important properties of normal division algebras, in particular the result that the direct product of a normal division algebra and its reciprocal is a total matrix algebra. Here one also finds the representation of the algebra as a direct sum of the radical and a sub-algebra isomorphic to the semi-simple difference algebra. Another chapter gives the theory of simple algebras and also the proof that every automorphism is an inner automorphism, together with properties of the maximal subfields and splitting fields. Next follows the theory of cross-products and factor sets giving the construction process for normal division algebras. After a study of the special cyclic semi-fields one obtains the theory of the important cyclic algebras and p-algebras. Very interesting is the chapter on rational division algebras which carries as far as to the proof of the theorem that every normal simple algebra over an algebraic field is cyclic. The remaining chapters contain theories in which the author's work has been of particular importance, namely, the theory of Riemann matrices and the theory of involutory algebras. In a final chapter miscellaneous results on various problems have been collected. *O. Ore* (New Haven, Conn.).

***Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band I. Algebra und Zahlentheorie. Teil 1. A. Grundlagen. B. Algebra. Heft 5. Hermes, Hans und Köthe, Gottfried. Theorie der Verbände.** 28 pp. B. G. Teubner, Leipzig, 1939.

The report on lattice theory for the new encyclopedia on algebra includes a clear exposition of the fundamental concepts and results of lattice theory. Precise statements of results and accurate bibliographical references are given for special questions, and for the applications of lattice theory to algebra, projective geometry, set theory and logic. No mention is made of partially ordered vector spaces, but a discussion of "regular rings" is included. *G. Birkhoff.*

Etherington, I. M. H. Genetic algebras. Proc. Roy. Soc. Edinburgh 59, 242–258 (1939). [MF 462]

Many significant features of genetic theory can be expressed mathematically in terms of linear algebras or hypercomplex number-systems, generally non-associative, over the rational or real fields; for example, if the units are taken

to represent different genetic types, their multiplication table represents a tabulation of the probable distribution among these types of the issues of the possible crosses between the several types. The simple instances previously noted by Glivenko [C. R. (Doklady) Acad. Sci. URSS **4**, 385–386 (1936)] are multiplied in the present paper so as to embrace a wide range of genetic situations. The author points out a number of algebraic properties and processes common to the several genetic algebras described, and proposes to discuss them elsewhere. In particular, he observes that the genetic algebras are all "special train algebras," that is, possess non-trivial homomorphic mappings on their coefficient fields and exhibit special features in respect to their nilpotent elements. *M. H. Stone.*

Jacobson, N. Cayley numbers and normal simple Lie algebras of type G. Duke Math. J. **5**, 775–783 (1939). [MF 817]

By the "derivation algebra" of a linear (not necessarily associative) algebra A is meant the set of linear operators D on A such that $(xy)D=x(yD)+(xD)y$. The author's principal result is that the different normal, simple Lie algebras of "type G" (in the author's classification of simple Lie algebras) over fields of characteristic zero are the "derivation algebras" of the different "Cayley systems" as obtained by M. Zorn [Abh. Math. Sem. Hans. Univ. **9**, 395–402 (1933)]. Moreover, the automorphisms of such a Lie algebra can be found if those of its defining Cayley system are known. This identifies two apparently different aspects of the theory of linear algebras. Some of the results had been indicated by Cartan for the case of algebraically complete fields. *G. Birkhoff* (Cambridge, Mass.).

MacDuffee, C. C. Modules and ideals in a Frobenius algebra. Monatsh. Math. Phys. **48**, 293–313 (1939). [MF 647]

Let $\mathfrak{A}$ be an associative algebra over the field $\mathfrak{F}$ with the basis $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$, multiplication being defined by $\epsilon_i\epsilon_j=\sum_k c_{ijk}\epsilon_k$, c_{ijk} in $\mathfrak{F}$. Then $\mathfrak{A}$ is a Frobenius algebra if the parastrophic matrix $Q=(\sum_r c_{ijr}l_r)$ with the row index i and the column index j is nonsingular for at least one set $l_1, l_2, \cdots, l_n$ of n elements of $\mathfrak{F}$. If $\alpha=a_1\epsilon_1+\cdots+a_n\epsilon_n$ (a_r in $\mathfrak{F}$) is an element of $\mathfrak{A}$, we form the regular representations $R(\alpha)=(\sum_r a_r c_{rji})$ and $S(\alpha)=(\sum_r a_r c_{ijr})$, where again i is the row index and j the column index. Every Frobenius algebra has a unit element 1, and the two regular representations are similar. Conversely, these two conditions characterize Frobenius algebras. We now assume that $\mathfrak{F}$ is the quotient field of a principal ideal ring $\mathfrak{R}$. Let $\mathfrak{J}$ be an integral domain of $\mathfrak{A}$ in the following sense: $\mathfrak{J}$ is a subring of $\mathfrak{A}$ of order n over $\mathfrak{R}$ which has an $\mathfrak{R}$-basis $\epsilon_1=1, \epsilon_2, \cdots, \epsilon_n$. If this basis is used, as we shall assume now, the c_{ijk} will lie in $\mathfrak{R}$. Every $\mathfrak{R}$-module $\mathfrak{M}$ in $\mathfrak{J}$ also has a linearly independent $\mathfrak{R}$-basis $\eta_1, \eta_2, \cdots, \eta_k$. We set $\eta_i=\sum_j a_{ij}\epsilon_j$ with a_{ij} in $\mathfrak{R}$. In this manner, modules can be described by matrices $\mathfrak{A}=(a_{ij})$. The formation of the union and of the intersection of two modules corresponds to the formation of a greatest common right divisor and of a least common left multiple of the associated matrices. For these operations with matrices, the methods of A. Chatelet [Groupes abeliens finis, Paris, 1924] and the author [Bull. Amer. Math. Soc. **39**, 564–584 (1933)] can be used. The product of two modules is studied from the same point of view. A necessary and sufficient condition that the matrix A correspond to a left ideal $\mathfrak{M}$ is that A be a common right divisor of $AR^T(\epsilon_1), \cdots, AR^T(\epsilon_n)$, that is, $AR^T(\epsilon_i)=D_i^TA$, where the D_i are matrices in $\mathfrak{R}$, the class matrices of A. An equivalent condition is that A be a common right divisor of the $S(\eta_i)$. If A corresponds to the left ideal $(\lambda_1, \cdots, \lambda_r|$, then A is a greatest common right divisor of $S(\lambda_1), \cdots, S(\lambda_r)$. The ideal matrices A which have the same class matrices D_i are gathered in a left minor class $\mathfrak{C}$; each such class is an $\mathfrak{R}$-module of matrices which has an $\mathfrak{R}$-basis consisting of at most n elements. We now restrict ourselves to the consideration of nonsingular left ideals which have the order n over $\mathfrak{R}$; the corresponding matrices A are nonsingular square matrices of degree n. Two such matrices A and B belong to the same minor class, if $A=BS(\beta)$, where β is an element of A which is not a divisor of zero. The matrix A is determined by the left ideal up to a left side factor U which is a matrix of degree n with coefficients in $\mathfrak{R}$ whose determinant is a unit. It then follows that two nonsingular left ideals belong to the same ideal class in the ordinary sense if and only if the corresponding matrices A and B satisfy a relation $A=UBS(\beta)$, where U and β satisfy the given conditions. An equivalent condition is that the class matrices of A and B be similar. Corresponding theorems can be given for right ideals.

R. Brauer (Toronto, Ont.).

Levitzki, Jakob. On rings which satisfy the minimum condition for the right-hand ideals. Compositio Math. **7**, 214–222 (1939). [MF 795]

The generalizations of the Wedderburn structure theorems for linear algebras lead to the consideration of rings A satisfying a minimal condition for left ideals (M L I); that is, each non-empty set of left ideals contains an ideal properly contained in no other ideal of the set. For these rings the behavior of the nilpotent elements is of major interest. One problem proposed by Köthe [Math. Z. **32**, 161–186 (1930)] and treated by Levitzki was recently solved by Charles Hopkins, who proved that in an M L I ring any subring consisting only of nilpotent elements is itself nilpotent [Duke Math. J. **4**, 664–667 (1938); Ann. of Math. **40**, 712–730 (1939)]. The present paper contains a new proof of Hopkins' result (submitted after publication of Hopkins' first paper). It contains also certain related theorems; for example, a non-nilpotent M L I ring A is a direct sum of a nilpotent left ideal and certain minimal potent left ideals. Levitzki's new method of proof rests essentially upon the assertion that the set of left hand annihilators of an M L I ring A can be characterized as the set of left hand annihilators of a suitable finite subset of A (Theorem 3). On this basis the theorem is reduced to a preliminary study of the nilpotency of subrings of A which are generated by finite subsets. Levitzki's proof is longer than Hopkins', but will later be applied to further classes of rings.

S. MacLane (Cambridge, Mass.).

Asano, K. Arithmetische Idealtheorie in nichtkommutativen Ringen. Jap. J. Math. **16**, 1–36 (1939). [MF 533]

The author considers rings $\mathfrak{S}$ in which every regular element (non-zero-divisor) has an inverse; $\mathfrak{o}$ is called an order if it is a subring containing the identity such that every element of $\mathfrak{S}$ has the form $a\alpha^{-1}$, a and α in $\mathfrak{o}$. The orders $\mathfrak{o}$ and $\mathfrak{o}'$ are equivalent if there exist regular elements λ, μ, λ', μ' such that $\lambda\mathfrak{o}\mu\subseteq\mathfrak{o}'$ and $\lambda'\mathfrak{o}'\mu'\subseteq\mathfrak{o}$; $\mathfrak{o}$ is maximal if it is contained in no equivalent order. The following assumptions are made: (I) There exists a maximal order bounded in the sense that for every regular α in $\mathfrak{o}$ there is a regular β such that $\mathfrak{o}\beta\subseteq\alpha\mathfrak{o}$. (II) The ascending chain con-

dition for integral two-sided ideals in $\mathfrak{o}$. (III) In each maximal order every (two-sided) prime ideal is strongly divisorless, that is, $\mathfrak{o}/\mathfrak{p}$ is a matrix ring over a quasi-field. On this basis the principal theorems of arithmetic in an algebra are derived and it is shown that in a certain sense these are necessary conditions for the theory. A notion of similarity (Gleichartigkeit) of ideals is discussed. Decomposition of normal ideals as a product of indecomposable factors is unique in the sense of similarity.

N. Jacobson (Chapel Hill, N. C.).

Iyanaga, S. Über die allgemeinen Hauptidealformeln. Monatsh. Math. Phys. **48**, 400–407 (1939). [MF 656]

Let Γ be a fixed finite Abelian group. The author calls a group $\mathfrak{G}$, with a finite set of generators, a Γ-group if $\mathfrak{G}$ has an Abelian invariant subgroup $\mathfrak{U}$ such that $\mathfrak{G}/\mathfrak{U}$ is isomorphic to Γ, and calls $\mathfrak{G}$ a complete (vollständig) Γ-group if this holds with $\mathfrak{U}=\mathfrak{G}'$, the commutator group of $\mathfrak{G}$. Let $S_1, S_\sigma, S_\tau, \cdots$ be representatives of the cosets of $\mathfrak{U}$ in $\mathfrak{G}$ corresponding to the elements $1, \sigma, \tau, \cdots$ of Γ, taking $S_1=1$ in $\mathfrak{G}$ and $S_\sigma, S_\tau, \cdots$ arbitrarily in their cosets. Then the principal ideal theorem of the class-field theory may be expressed as follows:

$$(1) \qquad \prod_{\sigma\Gamma} S_\sigma S_\tau S_{\sigma\tau}^{-1}=1, \qquad \text{for all } \tau\varepsilon\Gamma,$$

which hold for all complete Γ-groups $\mathfrak{G}$ and all choices of representatives $S_\sigma, S_\tau, \cdots$ [cf. Furtwängler, Abh. Math. Sem. Hans. Univ. **7**, 14–36 (1930); Iyanaga, ibid. **10**, 349–357 (1934)]. The author calls relations (1) principal ideal formulas and proves the following: An element Φ of the free group $\{S_\sigma, S_\tau, \cdots\}$ which has the value 1 for all complete Γ-groups $\mathfrak{G}$ is an element of the free group $\{\Psi^{-1}\Phi_1\Psi, \Psi^{-1}\Phi_2\Psi, \cdots\}$, where $\Phi_1, \Phi_2, \cdots$ are left members of relations (1), and Ψ is a fixed element of $\{S_\sigma, S_\tau, \cdots\}$. The proof of this result employs a Γ-group due to Magnus [J. Reine Angew. Math. **170**, 235–240 (1934)], which is here described more simply by means of the group ring: $R+R\sigma+R\tau+\cdots$, where R is the ring of rational integers.

R. Hull (Vancouver, B. C.).

Mori, Shinziro. Über die Produktzerlegung der Hauptideale. II. J. Sci. Hirosima Univ. Ser. A **9**, 145–155 (1939). [MF 669]

Let $\mathfrak{J}$ be an integral domain with unit for multiplication. The author investigates necessary and sufficient conditions in order that every principal ideal (a) of $\mathfrak{J}$ be a power product of a finite number of prime ideals (Property W =weak ideal theory in $\mathfrak{J}$). In order that W hold for $\mathfrak{J}$ it is necessary and sufficient that (I) every chain of ideal quotients $(a):\mathfrak{a}_1\subset(a):\mathfrak{a}_2\subset\cdots$ is finite, (II) every maximal prime ideal dividing (a) is invertible. For (II) one can substitute (II′): every prime ideal of $\mathfrak{J}$ contains a minimal prime ideal of $\mathfrak{J}$ and every minimal prime ideal of $\mathfrak{J}$ is invertible. The author employs the methods of classical ideal theory to obtain the complete description of integral domains having property W. *O. F. G. Schilling.*

Lorenzen, P. Abstrakte Begründung der multiplikativen Idealtheorie. Math. Z. **45**, 533–553 (1939). [MF 407]

The author studies the multiplicative theory of ideals through the consideration of semi-groups (Halbgruppen, ova) which are associative and commutative multiplicative systems. In this case it is also assumed that the cancellation law holds. Through an analysis and restatement of the conditions for evaluations and integral closure in the case of commutative rings, the author extends such concepts also to semi-groups and under suitable conditions many of the properties of the Dedekind-Prüfer ideal theory may be deduced in such systems. The author discusses also conjectures by Krull, disproves one and proves another under limiting assumptions. *O. Ore* (New Haven, Conn.).

Ritt, J. F. and Raudenbush, H. W., Jr. Ideal theory and algebraic difference equations. Trans. Amer. Math. Soc. **46**, 445–452 (1939). [MF 474]

The authors consider a difference ring in which a transform or difference operation is given. A difference ideal is invariant with respect to this transform, while a perfect difference ideal has the property that, if any product of transforms of an element belong to it, then the element itself belongs to the ideal. Every prime ideal is perfect. On the assumption of a certain basis condition every perfect ideal is the intersection of a finite set of prime ideals. Difference polynomials are introduced abstractly and it is shown that the difference ring generated by a finite set of difference variables satisfies the basis condition.

O. Ore (New Haven, Conn.).

Ritt, J. F. and Kolchin, E. R. On certain ideals of differential polynomials. Bull. Amer. Math. Soc. **45**, 895–898 (1939). [MF 778]

A set Σ of polynomials in the unknown y and its derivatives is an ideal if Σ is closed under differentiation, under addition, and under multiplication by any differential polynomial. The set of all solutions of $\Sigma=0$ (meromorphic in a suitable region) is known as the manifold $\mathfrak{M}$ of the ideal Σ. A decomposition theory for these ideals and their corresponding manifolds has been given by Raudenbush for the case of a "perfect" ideal Σ, such that G^m in Σ implies G in Σ [cf. J. F. Ritt, Semicentennial Publ. Amer. Math. Soc., vol. 2, pp. 35–55]. The present paper is part of an attack on the decomposition of an arbitrary ideal. The first step is the observation that, if $\mathfrak{M}$ is composed of two disjoint manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$, then Σ has a unique representation as a product of two ideals Σ_1 and Σ_2 with the respective manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ [J. F. Ritt, Proc. Nat. Acad. Sci. U. S. A. **25**, 90–91 (1939)]. The authors now analyse the structure of this ideal Σ_1 in terms of special assumptions on the manifold $\mathfrak{M}_1$. If $\mathfrak{M}_1$ consists of the solution $y=0$, and if Σ_1 contains some power y^p, while Σ contains a polynomial $F=y+A$, where all the terms of A are of degree at least 2 in y and its derivatives, the authors prove that Σ_1 actually contains y. This means that Σ_1 consists of all forms which vanish for $y=0$. The proof is obtained by repeatedly replacing the derivatives $y^{(i)}$ actually occurring in A by derivatives explicitly calculated from $y=F-A$. The result generalizes to n unknowns, but not to the case when F is y^m+A, with $m>1$. *S. MacLane* (Cambridge, Mass.).

Carlitz, L. Some sums involving polynomials in a Galois field. Duke Math. J. **5**, 941–947 (1939). [MF 829]

A primary polynomial with coefficients in a given finite field is one whose leading coefficient is unity. This paper deals with the evaluation of certain sums extended over the set of primary polynomials of given degree. Perhaps the simplest illustration is the sum of the reciprocals of all such polynomials of degree m; this sum turns out to be equal to

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the reciprocal of

$$(-1)^n(x^{p^{an}}-x)(x^{p^{a(n-1)}}-x)\cdots(x^{p^a}-x),$$

where p^a is the order of the field. Other sums evaluated are sums of certain powers of the polynomials or their reciprocals. *H. W. Brinkmann* (Swarthmore, Pa.).

Skolem, Th. Eine Bemerkung über gewisse Ringe mit Anwendung auf die Produktzerlegung von Polynomen. Norsk Mat. Tidsskr. **21**, 99–107 (1939). [MF 697]

Let R be a commutative ring with a unit element, and denote by N the set of those elements which are not divisors of zero. It is assumed that two elements a, b of N always have a greatest common divisor in N (that is, a common divisor which divides every common divisor). Further, the assumption is made that there are no infinite chains $a_1, a_2, a_3, \cdots$ in N such that every a_n is a proper divisor of the preceding a_{n-1}. Then the elements of N have a unique decomposition into prime factors. This is applied to the case of polynomials in n variables with coefficients in a given domain of integrity. *R. Brauer* (Toronto, Ont.).

Krull, Wolfgang. Funktionaldeterminanten und Diskriminanten bei Polynomen in mehreren Unbestimmten. Monatsh. Math. Phys. **48**, 353–368 (1939). [MF 652]

The discriminant $D(f_1, \cdots, f_n)$ of a set of n polynomials $f_i(x_1, \cdots, x_n)$ over an algebraically closed field is defined such that, if the equations $f_i=0$ have a finite number of solutions, then $D=0$ if and only if there exist multiple solutions by a suitable definition of multiplicity. The discriminant may be defined in various ways. The author gives a definition by means of resultants and functional determinants and proves various fundamental properties generalizing those of the case $n=1$. In particular, if the characteristic of the coefficient field is $p\neq 2$, the discriminant is an irreducible function of the coefficients. The author also proves a so-called functional determinant theorem which gives a generalization of the discriminant theorem for an arbitrary polynomial ideal, replacing roots and multiplicities by minimal prime ideals and isolated primary components. *O. Ore.*

MacLane, Saunders. The universality of formal power series fields. Bull. Amer. Math. Soc. **45**, 888–890 (1939). [MF 776]

Let $K=\{a\}$ be a field and $\Gamma=\{\alpha\}$ any ordered abelian group. Then the set S of all formal power series $a_{\alpha_1}t^{\alpha_1}+a_{\alpha_2}t^{\alpha_2}+\cdots+a_{\alpha_\rho}t^{\alpha_\rho}+\cdots$ (summed over all ordinals ρ less than a fixed σ, $\alpha_1<\alpha_2<\cdots<\alpha_\rho<\cdots$) is a field. If $n\gamma=\alpha$ is solvable in Γ for every pair α, γ in Γ and every integer n, then Γ is called a root group. The author shows that S is algebraically closed if K is algebraically closed and Γ is a root group. The proof follows by the Newton polygon method using the fact that S is a maximally complete field with respect to its natural valuation whose value group is Γ. Finally it is shown that under the same hypothesis on K and $\Gamma(\neq 0)$ the field S is universal, that is, it contains isomorphic images of every other field T which has the same cardinal number and characteristic as S. *O. F. G. Schilling* (Chicago, Ill.).

Zariski, Oscar. Algebraic varieties over ground fields of characteristic zero. Amer. J. Math. **62**, 187–221 (1940). [MF 968]

In a previous paper [Amer. J. Math. **61**, 249–294 (1939)] the author discussed properties of an algebraic variety over an algebraically closed field K of characteristic zero. The present paper extends to fields K which are not algebraically closed those results of the earlier paper which concern simple points of the variety. The first part of the paper is concerned with questions relating to the extension of a field K. Let K be a subfield of a field Σ of characteristic zero, and K^* a normal algebraic extension of K. A field Σ^* is constructed containing Σ and K^*. Let $\mathfrak{o}$ be a subring of Σ containing K and $\mathfrak{p}$ a prime ideal in $\mathfrak{o}$. There is a uniquely defined $\mathfrak{o}^*\subset\Sigma^*$, and at least one prime ideal $\mathfrak{p}^*$ in $\mathfrak{o}^*$ such that $\mathfrak{p}^*\cap\mathfrak{o}=\mathfrak{p}$; such a $\mathfrak{p}^*$ is said to lie over $\mathfrak{p}$. If $K_\mathfrak{p}$ is the residue class field of $\mathfrak{p}$, and $K^*_{\mathfrak{p}^*}$ of $\mathfrak{p}^*$, then $K^*_{\mathfrak{p}^*}$ coincides with the extension of $K_\mathfrak{p}$ obtained by extending K to K^*. Certain useful properties of the ideal $\mathfrak{o}^*\mathfrak{p}$ are also deduced.

Let Σ be a field of algebraic functions of r independent variables over K. With a set $\xi_1, \cdots, \xi_n$ of generators of Σ over K is associated an irreducible r-dimensional algebraic variety V_r whose general point has coordinates $\xi_1, \cdots, \xi_n$. With each s-dimensional prime ideal $\mathfrak{p}$ in $\mathfrak{o}=K[\xi_1, \cdots, \xi_n]$ is associated an irreducible s-dimensional subvariety V_s of V_r; an irreducible zero-dimensional subvariety is a point of V_r. Let $\mathfrak{J}$ be the quotient ring $\mathfrak{o}_\mathfrak{p}$, and $\mathfrak{P}$ the ideal $\mathfrak{J}\mathfrak{p}$; V_s is a simple subvariety of V_r if there exist $r-s$ elements $\eta_1, \cdots, \eta_{r-s}$ of $\mathfrak{J}$ such that $\mathfrak{J}\cdot(\eta_1, \cdots, \eta_{r-s})=\mathfrak{P}$. The η_i are said to be uniformizing parameters at V_s. The case $s=0$ is considered first. Let $\mathfrak{p}$ be zero-dimensional and determine a simple point P of V_r with $\eta_1, \cdots, \eta_r$ as uniformizing parameters. The treatment is first restricted to the case in which $K_\mathfrak{p}=K$; K^* is taken to be the algebraically closed field determined by K. Regarded as elements of Σ^* the elements $\xi_1, \cdots, \xi_n$ are the coordinates of the general point of a V_r^*. The ideal $\mathfrak{p}^*=\mathfrak{o}^*\mathfrak{p}$ is prime and zero-dimensional in $\mathfrak{o}^*$ and hence defines a point P^* of V_r^*; P^* is simple on V_r^* and $\eta_1, \cdots, \eta_r$ are uniformizing parameters at P^*. Hence the results of the previous paper can be applied to P^*. In this way the following theorems are proved.

A. There exist uniformizing parameters $\omega_1, \cdots, \omega_r$ at P which are elements of $\mathfrak{o}$ and are such that $\mathfrak{o}$ is integrally dependent on $K[\omega_1, \cdots, \omega_r]$.

B. Let $\eta_1, \cdots, \eta_r$ be algebraically independent elements of $\mathfrak{o}$ such that $\mathfrak{o}$ is integrally dependent on $K[\eta_1, \cdots, \eta_r]$. Let $\omega\subset\mathfrak{o}$, and let $G(\eta_1, \cdots, \eta_r; z)$ be the norm of $z-\omega$ with respect to $K(\eta_1, \cdots, \eta_r)$. Let $\eta_i\equiv c_i(\mathfrak{p})$, $c_i\subset K$. Then a necessary and sufficient condition that $\mathfrak{P}$ be a simple point with $\eta_1-c_1, \cdots, \eta_r-c_r$ as uniformizing parameters is that there exists an ω such that $G_\omega'(\eta_1, \cdots, \eta_r; \omega)\not\equiv 0(\mathfrak{p})$.

Passing to the case in which $K_\mathfrak{p}\neq K$, K^* is chosen to be the least normal extension of K which contains $K_\mathfrak{p}$. There are then a finite number of prime $\mathfrak{o}^*$-ideals $\mathfrak{p}_1^*, \cdots, \mathfrak{p}_n^*$, which lie over $\mathfrak{p}$. For each of these it is known that $K^*_{\mathfrak{p}_i^*}=K^*K_\mathfrak{p}=K^*$, and the preceding situation is obtained by showing that each corresponding P_i^* is simple and has the same uniformizing parameters as P. Extensions of A and B are thus obtained, as well as the following result:

C. $\mathfrak{J}$ is integrally closed in Σ, and contains the relative closure of K in Σ.

These properties of simple points are extended to simple subvarieties V_s of dimension $s>0$ by introducing a new ground-field $\Omega=K(\zeta_1, \cdots, \zeta_s)$, where $\zeta_1, \cdots, \zeta_s$ are elements of $\mathfrak{J}$ which are algebraically independent mod $\mathfrak{p}$. If $\tilde{\mathfrak{o}}=\Omega\mathfrak{o}$, $\tilde{\mathfrak{p}}=\tilde{\mathfrak{o}}\mathfrak{p}$, then $\tilde{\mathfrak{p}}$ is prime in $\tilde{\mathfrak{o}}$ and of dimension zero; $\xi_1, \cdots, \xi_n$ define a $\tilde{V}_{r-s}$ over Ω, and $\tilde{\mathfrak{p}}$ determines a point $\tilde{P}$ of $\tilde{V}_{r-s}$ which is simple if and only if V_s is simple. Extensions of A, B and C follow at once. *R. J. Walker.*

THEORY OF GROUPS

***Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band I. Algebra und Zahlentheorie. Teil 1. A. Grundlagen. B. Algebra. Heft 4. Teil 1. Magnus, Wilhelm. Allgemeine Gruppentheorie.** B. G. Teubner, Leipzig, 1939. 51 pp. RM 2.85.

The article on general group theory for the new algebra encyclopedia contains a thorough summary of the known facts about abstract groups. The discussion is condensed, but includes all relevant technical definitions, and a complete statement of most theorems. Little indication is given of methods of proof, but full references are made to the literature. Finite groups, free groups, group extension, infinite Abelian groups and topological groups are treated; permutation groups, collineation groups and special groups are not discussed. *G. Birkhoff* (Princeton, N. J.).

Miller, G. A. Independent generators of the subgroups of an Abelian group. Proc. Nat. Acad. Sci. U. S. A. **25**, 364–367 (1939). [MF 882]

In an Abelian group of order p^m it is found how many subgroups have given invariants. *J. S. Frame.*

Miller, G. A. Groups having a small number of subgroups. Proc. Nat. Acad. Sci. U. S. A. **25**, 367–371 (1939). [MF 883]

All groups are listed which separately have no more than seven proper subgroups. In counting the proper subgroups of a non-Abelian group G, the author considers how they may be permuted among themselves under G, noting that the permutation group must be intransitive and that its order must divide that of G. *J. S. Frame.*

Miller, G. A. Groups containing a prime number of non-invariant subgroups. Proc. Nat. Acad. Sci. U. S. A. **25**, 431–434 (1939). [MF 889]

The non-invariant subgroups of a group G are transformed under G according to a substitution group to which G is $\alpha, 1$ isomorphic. The properties of the invariant subgroup H of order α which corresponds to the identity substitution, and of the Sylow subgroups of G which correspond to transitive constituents of prime degree of the substitution group are first studied. It is then shown that if there are three or five non-invariant subgroups these must be permuted transitively. Finally the infinite systems of groups which separately contain a given prime number of non-invariant subgroups are determined.

J. S. Frame (Providence, R. I.).

Miller, G. A. Groups which contain less than ten proper subgroups. Proc. Nat. Acad. Sci. U. S. A. **25**, 482–485 (1939). [MF 881]

Twelve types of groups containing separately eight proper subgroups, and four types containing nine are listed, and it is shown that there are no others. *J. S. Frame.*

Miller, G. A. Groups having a small number of sets of conjugate subgroups. Proc. Nat. Acad. Sci. U. S. A. **25**, 640–643 (1939). [MF 727]

All groups are listed which contain exactly k proper complete sets of conjugate subgroups for $k=1, 2, 3, 4$, and those of certain types are listed for $k=5$. Besides the cyclic group of order p^{k+1} there are no others for $k=1$, and only the groups of order pq for $k=2$. For $k=3$ there are only the four group and the tetrahedral group, and for $k=4$ the quaternion group, the non-cyclic group of order 9 and certain groups of order p^2q and p^3q, where p and q are distinct primes satisfying given conditions. *J. S. Frame.*

Derry, Douglas. On finite abelian p-groups. Bull. Amer. Math. Soc. **45**, 874–881 (1939). [MF 773]

Here $\mathfrak{G}$ denotes an abelian group whose order is some power of a prime p. A set of elements $S_1, \cdots, S_n$ of $\mathfrak{G}$ is said to define a composition series for $\mathfrak{G}$ if the group series

$$(S_1, \cdots, S_n), \quad (S_1, \cdots, S_{n-1}), \quad \cdots, \quad (S_1, S_2), \quad (S_1), \quad (E)$$

is a composition series for $\mathfrak{G}$. A set of elements $S_1, \cdots, S_n$ of $\mathfrak{G}$ is said to form a coverage system of $\mathfrak{G}$ if, for every subgroup $\mathfrak{H}$ whose quotient group $\mathfrak{G}/\mathfrak{H}$ is cyclic, we may select elements $S_{m_1}, \cdots, S_{m_k}$ so that the group series

$$(S_{m_1}, \cdots, S_{m_k}, \mathfrak{H}), \quad (S_{m_1}, \cdots, S_{m_{k-1}}, \mathfrak{H}), \quad \cdots, \quad (S_{m_1}, \mathfrak{H}), \quad \mathfrak{H}$$

is a composition series from $\mathfrak{G}$ to $\mathfrak{H}$. After remarking that a set of elements which defines a composition series for $\mathfrak{G}$ also forms a coverage system for $\mathfrak{G}$, the author proves the following partial converse: A coverage system $S_1, \cdots, S_n$ for an abelian p-group $\mathfrak{G}$ of rank 2 with composition series of length $n+1$ defines a composition series for $\mathfrak{G}$ after a possible rearrangement of order. (The corresponding result for rank 1 is immediate.) *H. S. M. Coxeter.*

Derry, D. Remarks on a conjecture of Minkowski. Amer. J. Math. **62**, 61–66 (1940). [MF 955]

The author shows that a special case of Minkowski's hypothesis "$M\,I$" on linear forms [Mordell, C. R. Congr. Int. Math. Oslo, vol. 1, 226–238 (1936)] can be expressed in group-theoretic terms as follows. Let $\mathfrak{B}$ be an Abelian p-group of order p^m with rank less than n, and let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n$ be cyclic subgroups having the following property: for every subgroup $\mathfrak{D}$ whose factor group $\mathfrak{B}/\mathfrak{D}$ is cyclic, groups $\mathfrak{C}_{s_1}, \mathfrak{C}_{s_2}, \cdots, \mathfrak{C}_{s_k}$ exist such that the factors of the series

$$\mathfrak{B}=(\mathfrak{C}_{s_1}, \mathfrak{C}_{s_2}, \cdots, \mathfrak{C}_{s_k}, \mathfrak{D}), \quad \cdots, \quad (\mathfrak{C}_{s_1}, \mathfrak{D}), \quad \mathfrak{D}$$

have order not greater than p^r. Then (according to the conjecture) the groups $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n$, after a possible rearrangement, build a series

$$\mathfrak{B}=(\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_n), \quad \cdots, \quad (\mathfrak{C}_1, \mathfrak{C}_2), \quad \mathfrak{C}_1, \quad (E),$$

whose factors are all cyclic and of order p^r.

H. S. M. Coxeter (Toronto, Ont.).

Asano, K. Über verallgemeinerte Abelsche Gruppe mit hyperkomplexem Operatorenring und ihre Anwendungen. Jap. J. Math. **15**, 231–253 (1939). [MF 532]

The author proves that a ring in which both chain conditions hold is a principal ideal ring if and only if it is einreihig in the sense of Köthe [Math. Z. **39**, 31 (1934)]. He then discusses Abelian groups whose operators form an einreihig ring and their rings of operator automorphisms. Using operator methods he extends a number of results well-known for semi-simple rings to the case of einreihig rings. For example, if $\mathfrak{a}$ over K is an einreihig subalgebra of a simple ring and $V(\mathfrak{a})$ the set of elements commutative with the elements of $\mathfrak{a}$ then $V(V(\mathfrak{a}))=\mathfrak{a}$. *N. Jacobson.*

Bernstein, B. A. Groups and Abelian groups in terms of negative addition and negation. Duke Math. J. **5**, 871–874 (1939). [MF 826]

The following postulates are given in terms of the operations a' (the inverse of a), and $a\Delta b$, which is to be the negative $(a+b)'$ of group addition. P_0: K contains at least two distinct elements. P_1: $a\Delta b$ is in K. P_2: a' is in K. P_3: $(a\Delta b)'\Delta c = a\Delta(b\Delta c)'$. P_4: $(a\Delta b)\Delta a = b$. P_5: $(a\Delta b)' = b'\Delta a'$. P_6: $a\Delta b = b\Delta a$, whenever the elements, and in P_5–P_6 their combinations, are in K. It is shown that P_0–P_5 are independent, that P_0–P_5 are sufficient for groups, and P_0–P_6 for Abelian groups. Additional theorems are given involving the operation Δ which are of interest in themselves, but are not needed to establish the sufficiency of the postulate system. *O. Frink* (State College, Pa.).

Easterfield, T. E. The orders of products and commutators in prime-power groups. Proc. Cambridge Philos. Soc. **36**, 14–26 (1940). [MF 895]

P. Hall in his fundamental paper on p-groups [Proc. London Math. Soc. **36**, 29–95 (1934)] proved that, in a regular p-group G, the order of the product of elements cannot exceed the order of all the factors; a group G is certainly regular if its class c is less than p. The author considers an arbitrary p-group G of class c. If P and Q are two elements of order p^α, p^β, respectively, then he proves $(PQ)^{p^x} = P^{p^x}$, if

$$x \geqq \beta + \left[\frac{c-1}{p-1}\right],$$

where $[a]$ denotes the largest integer $n \leqq a$. In particular, the order of PQ is at most p^m with

$$m = \max\left(\alpha, \beta + \left[\frac{c-1}{p-1}\right]\right).$$

If the commutator (P, Q) is of order p^r, then the order of P relative to the central of $\{P, Q\}$ cannot exceed p^x, where

$$x = r + \left[\frac{c-2}{2(p-1)} + \frac{1}{2}\right].$$

For $c \leqq p-1$ (and hence $x = r$) this had been proved by Hall. Further, commutators R of weight ρ are considered which contain a commutator R_1 of weight ρ_1. If R_1 has the order p^{r_1} relative to the $(\rho-1)$th central $Z_{\rho-1}$ of G, then the order of R cannot exceed p^x, where

$$x = r_1 + \left[\frac{c-\rho}{\rho_1(p-1)}\right].$$

Certain cases are discussed in which c can be replaced by a smaller integer. On the other hand, examples are constructed in which the inequalities cannot be improved. The proof depends upon two expansion theorems similar to that given by Hall. The first theorem deals with an expansion of (Q^{p^s}, P^{p^s}), the second with the powers R^{p^x} of a given commutator R which contains a given commutator R_1.

R. Brauer (Toronto, Ont.).

Scholz, Arnold. Abelsche Durchkreuzung. Monatsh. Math. Phys. **48**, 340–352 (1939). [MF 651]

The author considers extensions G of (normal) subgroups N with the property that N is part of both the central and the commutator-subgroup of G. Two such groups belong to the same class if they may be represented as quotient-groups H/N_1 and H/N_2 of a group H containing a normal subgroup M so that $N_1M = MN_2$, $N_1 \cap M = 1 = M \cap N_2$, and so that H/M is Abelian. (Note that N_1N_2 is part of both the central and the commutator-subgroup of H/N_i.) A method is given for deriving all the groups in a class from one of these. The number of groups in the class, containing G with the distinguished normal subgroup N, is the product of all the g.c.d.'s (n, m), where n is any invariant of the Abelian group N, m any invariant of G mod its commutator subgroup. *R. Baer* (Urbana, Ill.).

Ore, Oystein. A remark on groups which are the direct product of their Sylow groups. Monatsh. Math. Phys. **48**, 41–42 (1939). [MF 624]

These groups, called nilpotent groups, are shown in the main theorem to be the only groups in which every subgroup is a composition group. The terms principal chain, maximal principal chain, composition chain, and maximal composition chain of subgroups are first defined, and a subgroup is called a composition group if it occurs in some composition chain. Two preliminary theorems about the structural relations of composition groups, and two more about composition chains in p-groups lead to the main theorem. *J. S. Frame* (Providence, R. I.).

Hoheisel, Guido. Über Charaktere. Monatsh. Math. Phys. **48**, 448–456 (1939). [MF 661]

Instead of considering the individual operations and their representative matrices in building the theory of the characters of a finite group, the author bases this theory directly upon the multiplication table of the conjugate sets G_α. Corresponding to the relations $G_\alpha G_\beta = \sum c_{\alpha\beta\gamma} G_\gamma$, he considers the equations

$$(1) \qquad x_\alpha \cdot x_\beta = \sum_{\gamma=1}^{n} c_{\alpha\beta\gamma} x_\gamma,$$

in which the $c_{\alpha\beta\gamma}$ satisfy appropriate conditions, namely: (i) $\sum c_{\alpha\rho\gamma} c_{\gamma\tau\delta} = \sum c_{\alpha\tau\gamma} c_{\gamma\rho\delta}$, (ii) $c_{1\alpha\beta} = c_{\alpha'1\beta'}$, (iii) $c_{\alpha\beta1} = 0$ for $\alpha \neq \beta'$ or h_α for $\alpha = \beta'$, where $h_\alpha > 0$ and $h_1 = 1$. Condition (i) corresponds to the associative-commutative relation $(G_\alpha G_\mu) G_\nu = (G_\alpha G_\nu) G_\mu$. For such a set of c's there are n solutions $(x_{1,k}, x_{2,k}, \cdots, x_{n,k})$ of (1), where

$$x_{\alpha,k} \cdot x_{\beta,k} = \sum_{\gamma=1}^{n} c_{\alpha\beta\gamma} x_{\gamma,k}, \qquad k = 1, 2, \cdots, n,$$

and these solutions are linearly independent. It follows from the proof of this theorem that $x_{\alpha,k} = x_{\alpha',k}$ and that $\alpha = \alpha'$ if the $x_{\alpha,k}$ are all real, for any given k. The n equations $\sum g_k x_{\gamma,k} = 0$ for $\gamma \neq 1$ or h for $\gamma = 1$, where $h = \sum_{\alpha=1}^{n} h_\alpha$, determine n numbers g_k which may be shown to be positive. If we write $g_k = f_k^2$, we obtain the characters by setting

$$\chi_\alpha^{(k)} = \frac{f_k}{h_\alpha} x_{\alpha,k},$$

from which all the familiar relations follow immediately [cf. Speiser, Gruppentheorie, 3rd ed., § 57].

G. de B. Robinson (Toronto, Ont.).

Baer, Reinhold. Duality and commutativity of groups. Duke Math. J. **5**, 824–838 (1939). [MF 823]

A (1-1) correspondence between the subgroup lattices of two groups is called a subgroup isomorphism if the relations $S \leqq T$ are preserved, and a dualism if they are reversed. (If, in the sequel, a group is denoted by G, it shall be understood that it admits a dualism.) If a group G is subgroup isomorphic with H, then clearly H is a dual of G; the prin-

cipal result of the paper is the converse of this in case G is abelian. In other words, it is maintained that an abelian G is dual to itself. The proof is based on the theory of infinite abelian groups and on the following theorems. (a) Every quotient group of G has a dual. (b) Every element of a group G is of finite order. (c) If all elements of an abelian group G except one have the same order, then the group is finite. From these theorems and several structure properties of abelian groups it follows that the abelian group G has only elements of finite order, and that there is only a finite number of elements whose orders are powers of the same prime. But an abelian group of this type is self-dual, according to a previous result [Baer: Dualism in abelian groups, Bull. Amer. Math. Soc. **43**, 121–124 (1937)]; in fact the autodualism may be chosen such that if T corresponds to S, the quotient groups G/T and G/S are respectively isomorphic to S and T. The paper concludes with an appendix, in which an interesting result about the dualism between an abelian group G and its character group is announced. *M. A. Zorn* (Los Angeles, Calif.).

Shaw, James Byrnie. Algebras defined by groups whose members are of the form A^aB^ν. Duke Math. J. **5**, 839–855 (1939). [MF 824]

The problem considered in this paper is that of reducing the group algebra over the complex field of groups whose elements have the form A^aB^ν, where $A^\alpha=1=B^\lambda$ and where $BA=A^qB^r$. The use of a number of unusual terms makes it rather difficult to read. *N. Jacobson.*

Vandiver, H. S. On the imbedding of one semi-group in another, with application to semi-rings. Amer. J. Math. **62**, 72–78 (1940). [MF 957]

Let G be any system in which an associative multiplication is defined. It is obvious that for G to be extensible to a group, it is necessary that the laws of cancellation hold. It is well known that if G is commutative, this is sufficient, and known that this is not always sufficient if G is non-commutative. The author shows that if the cancellable elements commute with all elements, then the cancellable part of G (which is always closed under multiplication) can be extended to a group in a way including the rest of G. A similar result is proved for rings. *G. Birkhoff.*

Eaton, J. E. and Ore, Oystein. Remarks on multigroups. Amer. J. Math. **62**, 67–71 (1940). [MF 956]

The authors show that, if $\mathfrak{A}$ is a left reversible and $\mathfrak{B}$ a left closed submultigroup of $\mathfrak{M}$, then the cross-cut $(\mathfrak{A}, \mathfrak{B})$ is not void; and, if in addition $\mathfrak{A}$ is normal, then every element of the union $[\mathfrak{A}, \mathfrak{B}]$ is in some product ab. With the aid of these theorems, the inheritance by $(\mathfrak{A}, \mathfrak{B})$ and $[\mathfrak{A}, \mathfrak{B}]$ of certain properties of $\mathfrak{A}$ and $\mathfrak{B}$ is established without requiring a unit in $\mathfrak{M}$, thus improving a result in chapter 3, section 2 of a paper by Dresher and Ore [Amer. J. Math. **60**, 705–733 (1938)]. Definitions of homomorphism and normality as given in the above cited paper are modified in order to correct in one case, and to strengthen in the other, certain theorems of Dresher and Ore.
H. S. Wall (Evanston, Ill.).

Eaton, J. E. Associative multiplicative systems. Amer. J. Math. **62**, 222–232 (1940). [MF 969]

The author considers an associative multiplicative system in which the product of any two elements is a set of elements, but which differs from a hypergroup or multigroup in that there is no quotient postulate. He shows that there is a left (right) coset decomposition in terms of any left (right) reversible subsystem A, where he defines A to be left reversible if when $Ax \supset y$ then $Ay \supset x$, and right reversible is similarly defined. Homomorphisms generated by such decompositions are characterized. A Jordan-Hölder theorem is given for certain normal subsets. Some of the results extend or improve results obtained by Dresher and Ore [Amer. J. Math. **60**, 705–733 (1938)]. *H. S. Wall.*

TOPOLOGY

Vázsonyi, Endre. Über Gitterpunkte des mehrdimensionalen Raumes. Acta Litt. Sci. Szeged **9**, 163–173 (1939). [MF 608]

Consider the graph G formed by the integral points in n-space, each point (vertex) p being joined to each p' which is obtained from p by changing just one coordinate by 1. It is shown that there is an arc in G, going to ∞ in both directions, which passes over each vertex just once; also one which passes over each arc just once. It is also shown that the vertices may be covered, each just once, by a path composed of knight's moves. The proofs use induction from the 2-dimensional case in a neat fashion. *H. Whitney.*

Tucker, A. W. On chain-mappings carried by cell-mappings. Proc. Nat. Acad. Sci. U. S. A. **25**, 371–374 (1939). [MF 884]

Let X, X' be abstract complexes; let C and F denote closure and boundary operators. Cell- and chain-mappings S and T of X into X' are determined by letting correspond to each cell x a set of cells Sx, and to each oriented x, a chain Tx, in X', such that $CSx=CSCx$ and $FTx=TFx$; T is carried by S if Tx is in Sx; T is equivalent to T_1 over S if D exists, Dx in Sx, such that $FDx=Tx-T_1x-DFx$. A main theorem is that if cycles of each Sx bound, then any two T, T_1 carried by S are equivalent; they then give the same transformations of the homology groups. Applications to subdivisions, graphs and products are given.
H. Whitney (Cambridge, Mass.).

Tucker, A. W. The algebraic structure of complexes. Proc. Nat. Acad. Sci. U. S. A. **25**, 643–647 (1939). [MF 728]

A complex is here considered as a set of cells on which two matrices E, F operate; F is the boundary operator, while E merely reorients cells of odd dimension, in the natural interpretation. A mapping T of a complex X into another X' is a "chain mapping" if $EFTx=TEFx$ for all x. The theory before developed [Proc. Nat. Acad. Sci. U. S. A. **25**, 371–374 (1939)] is extended to the present situation. "Lefschetz numbers" and products are also considered.
H. Whitney (Cambridge, Mass.).

Reidemeister, Kurt. Durchschnitt und Schnitt von Homotopieketten. Monatsh. Math. Phys. **48**, 226–239 (1939). [MF 641]

The author defines a new type of intersection of the homotopy chains and cycles [for terminology see K. Reidemeister, Topologie der Polyeder, Leipzig, 1938, § 17] of a manifold with the usual properties that the intersection of

two cycles is a cycle and if either bounds so does the intersection. The new intersection is not however a homotopy chain but is a linear form in the elements of the fundamental group with coefficients which are ordinary intersections of the first homotopy chain with translations of the second. He defines simplified intersections by mapping the coefficient homotopy chains into ordinary chains. Both types of intersection are defined also for the chains of the generalized "coverings" (Überdeckungen) belonging to the right ideals of the group-ring of the fundamental group. Using the simplified intersections he gives generalizations of intersection numbers and linking numbers. *C. H. Dowker.*

Flexner, William W. The generality of finite abstract complexes. Proc. Nat. Acad. Sci. U. S. A. **25**, 637–639 (1939). [MF 726]

This note shows that the homology groups of a finite abstract complex are no more general than those of a simplicial complex. Given any finite abstract complex A, that is, any finite collection of "cells" E^p with dimensions p and incidence numbers $[E^p : E^{p-1}]$ such that $\sum[E^{p+1} : E^p][E^p : E^{p-1}]=0$, the author constructs a simplicial complex containing an open subcomplex B whose p-dimensional homology group is isomorphic with the $p+k$-dimensional homology group of A for all p, k being a constant ($\geqq 0$) which arises in the construction.

A. W. Tucker (Princeton, N. J.).

Komatu, A. und Sakata, R. Über ein Problem von Herrn Borsuk. Jap. J. Math. **16**, 57–62 (1939). [MF 535]

It has been proved by K. Borsuk [Fund. Math. **28**, 203–210] that a mapping of an acyclic polytope (that is, a polytope whose homology groups of all dimensions are vacuous) on a spherical surface of arbitrary dimension is unessential. The authors prove the converse theorem: A polytope which is not acyclic can be mapped essentially on a sphere of some dimension. The same result has been proved by Eilenberg [Generalisation du théorème de M. Hopf, Compositio Math. **6** (1939)]. *W. Hurewicz.*

Kodaira, K. Die Kuratowskische Abbildung und der Hopfsche Erweiterungssatz. Compositio Math. **7**, 177–184 (1939). [MF 791]

Abe, Makoto. Über die Methode der Polyederentwicklung der Kompakten und ihre Anwendungen auf die Abbildungstheorie. Compositio Math. **7**, 185–193 (1939). [MF 792]

Kodaira gives an explicit proof of Hopf's extension theorem [Alexandroff-Hopf, Topologie I, 1935, p. 501, Satz II], replacing complex by compact metric space and using the additive group $\mathfrak{R}$ of reals mod 1 as coefficients. This result is implicitly contained in the paper of Alexandroff, Hopf, and Pontrjagin, Compositio Math. **4**, 237–255 (1937). Abe gives another proof of this theorem, with $\mathfrak{R}$ replaced by the group $\mathfrak{R}_1$ of rationals mod 1. He also gives another proof of Hopf's theorem on the classification of mappings in spheres [Alexandroff-Hopf, loc. cit., p. 509, Satz I, and Freudenthal, Compositio Math. **2**, 134–162 (1935)], using $\mathfrak{R}_1$ as coefficient group instead of $\mathfrak{R}$. *H. Wallman.*

Freudenthal, Hans. Die Triangulation der differenzierbaren Mannigfaltigkeiten. Nederl. Akad. Wetensch., Proc. **42**, 880–901 (1939). [MF 705]

A proof is given that every differentiable manifold of class C^r can be triangulated (with cells of class C^r). The theorem is due to S. S. Cairns [see Ann. of Math. **35**, 579–587 (1934) and Bull. Amer. Math. Soc. **41**, 549–552 (1935)].

H. Whitney (Cambridge, Mass.).

Pitcher, Everett. Identification of two subsets. Proc. Nat. Acad. Sci. U. S. A. **25**, 435–438 (1939). [MF 890]

Let K be a complex, C_1 and C_2 two sub-complexes separated in K and each a copy of a complex C; C_1 and C_2 are identified, yielding a new complex L; $B^k(K)$ denotes the k-dimensional homology group of K. The coefficient group is arbitrary; S_k denotes the subgroup of $B^k(L)$ consisting of classes of k-cycles on L which are images of k-cycles on K; T^k denotes the group of the classes of k-cycles on K whose images bound on L; $N^k(C)$ denotes the group of the classes of k-cycles on C for each of which the difference of the copies on C_1 and C_2 bounds on K. The principal theorem states that the following three isomorphisms are valid: (1) $B^k(L) \bmod S^k(L) \approx N^{k-1}(C)$; (2) $B^k(K) \bmod T^k \approx S^k$; (3) $B^k(C) \bmod N^k(C) \approx T^k$ ($k=0, 1, \cdots, n$). Here $N^{-1}(C)$ denotes the group with one element. (In the notation of Alexandroff-Hopf, Topologie, vol. 1, "mod" would be replaced by the minus sign.) The proof is given in outline. The following corollary is given: Let $p^k(K)$ and n^k denote the ranks [Alexandroff-Hopf, p. 563] of $B^k(K)$ and $N^k(C)$, respectively. Then $p^k(L)=p^k(K)-p^k(C)+n^k+n^{k-1}$. If K consists of two separated parts, one containing C_1 and the other C_2, the corollary is the so-called Mayer-Vietoris formula [W. Mayer, Monatsh. Math. Phys. **36**, 1–42 (1929); L. Vietoris, ibid. **37**, 150–162 (1930); Alexandroff-Hopf, p. 299]. Four extensions are given: (1) case that C_1 and C_2 are not separated; (2) case of singular chains; (3) case of Vietoris cycles with coefficients from a field; (4) case of cohomology groups, in which $k-1$ is replaced by $k+1$ in the relations.

The author states that "A. B. Brown [Proc. Nat. Acad. Sci. U. S. A. **16**, 401–406 (1930)] has obtained relations for a complex which imply our corollary for some non-negative integers n^k." The "some" should be changed to "all." The restriction in the work of Brown is in the coefficient group. The first statement of Brown's theorem was for the case that the two complexes C_1 and C_2 of the present paper are replaced by any finite number of complexes. The Mayer-Vietoris formula was also derived independently by Brown [Amer. J. Math. **52**, 253 (1930); add (4.1) and (4.2) to obtain the formula (β_i defined on p. 257)]. These relations (4.1) and (4.2) were used to obtain relations between critical points, for which the Mayer-Vietoris formula alone is insufficient. *A. B. Brown* (Flushing, N. Y.).

Riesz, Frédéric. Sur le théorème de Jordan. Acta Litt. Sci. Szeged **9**, 154–162 (1939). [MF 607]

In many of the proofs which have been given of the Jordan curve theorem, the notion of order of a point not on the curve has been the basic tool. In the first half of this paper it is shown that the proof of the theorem can be carried out by classifying the points not on the curve according to the parity of their order without being concerned with the actual value of the order, and subsequently reducing the resulting two cases to one case by an inversion. In the second half, problems concerning the separation of the plane by (1) a general closed set and (2) a continuous curve are handled by use of the order of points relative to polygons with sufficiently small edges whose vertices lie in the set. Here close contact is made with ideas already in use in

topology, although the paper as a whole is readily understandable by readers not acquainted with the field.

R. L. Wilder (Ann Arbor, Mich.).

Moore, R. L. Concerning the open subsets of a plane continuum. Proc. Nat. Acad. Sci. U. S. A. **26**, 24–25 (1940). [MF 999]

It is shown that, if a continuum M in the plane is the sum of a closed set K and the elements of a countable collection G of disjoint continua no one of which intersects K, every element of G is a component of $M-K$. Also, there exists a plane continuum M containing a closed set K such that $M-K$ has only a countable number of components and K contains no limit point of any one of them.

G. T. Whyburn (Charlottesville, Va.).

Moore, R. L. Concerning accessibility. Proc. Nat. Acad. Sci. U. S. A. **25**, 648–653 (1939). [MF 729]

A study is made of the proposition: "(A) If the boundary of a connected domain D is a compact set B, there exists a compact continuum containing B and lying in $D+B$," in connection with axioms 0, 1 and 2 of the author's book "Foundations of Point Set Theory" and the following axioms: (F) If P is a point of a region R and X is a point distinct from P, there exists a finite set lying in R and separating P from X; (K) If P is a point of a region R, there exists in R a domain D containing P and bounded by a compact set. It is shown, for example, that axiom (K), but not axiom (F), suffices in the presence of axioms 0, 1 and 2 to yield proposition (A). The same axioms yield the conclusion that any continuum with a compact boundary is compactly connected and also numerous other results.

G. T. Whyburn (Charlottesville, Va.).

Cooper, R. Transformations of enumerable sets which are dense in an interval. Quart. J. Math., Oxford Ser. **10**, 247–251 (1939). [MF 1034]

This is concerned with 1-1 bicontinuous transformations which may be set up between two enumerable sets E and F dense in respective, not necessarily bounded, linear intervals. In particular, when E and F are dense in the interval $(0, 1)$, a function may be constructed which is defined everywhere on that interval, is convex and increasing, and possesses derivatives to a preassigned order; the range of the first derivative may be taken arbitrarily close to 1 and that of all higher derivatives arbitrarily close to 0.

L. Zippin (Flushing, N. Y.).

Hall, Dick Wick. An example in the theory of pointwise periodic homeomorphisms. Bull. Amer. Math. Soc. **45**, 882–885 (1939). [MF 774]

In the usual examples of pointwise periodic transformations every convergent sequence of orbits converges to a set of points all of which have the same period. The first example showing that this need not be true was given recently by Schweigert and the author [Duke Math. J. **4**, 723 (1938)]. The present paper combines the ideas of this example with one constructed by Phillips and Ayres [Fund. Math. **33**, 101 (1939)] to give an example where the space is locally connected and a sequence of orbits converges to a set containing points of arbitrary preassigned periods.

W. L. Ayres (Ann Arbor, Mich.).

Pitcher, Everett. Critical points of a map to a circle. Proc. Nat. Acad. Sci. U. S. A. **25**, 428–431 (1939). [MF 888]

Morse's well-known work on critical points gives relations between the number of critical points of a function defined on a space and the homology classes of the space. A real-valued function on a space S can be considered as a map of S on the straight line. The author generalizes to a study of the critical points of a map of a manifold M on a circle. Introducing a covering space K of M induced by the map and using definitions and methods similar to those of Morse, relations are obtained between the count of critical points of the map on the circle and the ranks of certain homology groups on M. These ranks depend not only on M, but also on the homotopy class of the map, and are invariants of the latter.

S. B. Myers (Ann Arbor, Mich.).

Yamauti, Syôzô. Zwei Bemerkungen über den Produktraum. Proc. Phys.-Math. Soc. Japan **21**, 609–614 (1939). [MF 1125]

Given a compact subset F of a euclidean R^m, let $u_J{}^r(F)$ be the Urysohn-Alexandroff constant of F relative to the dimension r and the coefficient group J [Alexandroff, Fund. Math. **20**, 140–150 (1933)]. The author proves the following inequality concerning cartesian products:

$$u_J{}^{p+1}(F_1\times F_2)\geqq\max\,[\min\,(u_J{}^{n_1+1}(F_1),\ u_J{}^{n_2+1}(F_2))],$$

where $F_1\subset R^{n_1}$, $F_2\subset R^{n_2}$, $p=n_1+n_2$ and J is a ring. The second remark brings the proof that $\pi_n(R_1\times R_2)$ is the direct product of $\pi_n(R_1)$ and $\pi_n(R_2)$, where $\pi_n(R)$ is Hurewicz's nth homotopy group of the metric space R.

S. Eilenberg (Ann Arbor, Mich.).

Vedenissoff, N. Généralisation de quelques théorèmes sur la dimension. Compositio Math. **7**, 194–200 (1939). [MF 793]

This paper defines dimension for general topological spaces in terms of finite coverings by open sets and then proves that certain theorems, familiar for simpler spaces, remain true for more general ones. Let $\omega=\{O_0, O_1, \cdots, O_s\}$ be a covering of a space X. A continuous transformation of X into Y by a function $f(x)$ is called an ω-transformation if for every y in $f(X)$ the set $f^{-1}(y)$ is contained in some set of ω. The principal theorem is the following: Let X be a compact perfectly normal space. In order that the dimension of X be n it is necessary and sufficient (1) that to every covering ω there is an ω-transformation of X into an n-dimensional polyhedron and (2) that there is some covering $\tilde{\omega}$ of X for which there is no $\tilde{\omega}$-transformation of X into a polyhedron of dimension less than n.

D. Montgomery (Northampton, Mass.).

Cartan, H. et Dieudonné, J. Notes de tératopologie. III. Rev. Sci. (Rev. Rose Illus.) **77**, 413–414 (1939). [MF 459]

Les auteurs munissent le groupe d'addition des nombres réels d'une topologie plus fine que sa topologie ordinaire (et quand même satisfaisant le deuxième axiome de dénombrabilité) pour réfuter un théorème qui se trouve chez le rapporteur [Ann. of Math. **37**, 46–56 (1936), no. 7]. Le rapporteur regrette d'avoir omis dans le texte de ce théorème les mots "im Kleinen kompakt" devant "Gruppe," mots indispensables pour la validité de la démonstration.

Une topologie du même groupe, mais moins fine que la topologie ordinaire sert aux auteurs d'exemple d'un espace F (de Banach), où λx est fonction continue de λ et de x, mais non du couple λ, x. D'après un théorème de M. Mazur cité par les auteurs, un tel espace n'est point métrisable. Un autre exemple concerne l'essai, fait par Bourbaki [C. R. Acad. Sci. Paris **206**, 1701–1704 (1938)], de baser la topo-

logie faible des espaces linéaires sur la théorie des filtres de M. Cartan. L'exemple fait voir que cet essai n'a pas absolument réussi: un espace normé ne peut être faiblement complet que s'il a un nombre fini de dimensions.

Enfin les auteurs démontrent l'inexactitude d'une des affirmations se trouvant dans la note citée de Bourbaki.

H. Freudenthal (Amsterdam).

Dieudonné, Jean. Sur les espaces topologiques susceptibles d'être munis d'une structure uniforme d'espace complet. C. R. Acad. Sci. Paris **209**, 666–668 (1939). [MF 710]

Every space, which admits a uniform structure, and whose topology is finer than that of a metric space, admits a uniform structure in which it is complete. The author remarks that his example [these Rev. **1**, 30 (1940)] of a space admitting no complete uniform structure is included in a general class of such examples. This follows from a theorem of A. Weil. He also remarks that this example leads to an example of two normal spaces whose product is not normal. (The same example and a different proof were discovered, but not yet published, by E. Čech.)

J. W. Tukey (Princeton, N. J.).

Rothe, Erich. The theory of topological order in some linear topological spaces. Iowa State Coll. J. Sci. **13**, 373–390 (1939). [MF 495]

This paper presents extensions of previous results of the author [Compositio Math. **4**, 294–307 (1936–37); **5**, 177–197 (1937–38)] concerning the generalization, to linear normed (or normable) spaces, of the notion of the order of a point in n-space with respect to an $(n-1)$-dimensional cycle. Let E be a linear normed space, V a closed convex bounded subset with inner points, S its boundary and $f(x)$ a transformation on S into E, supposed to be of the form $f(x)=\lambda(x)\cdot x+F(x)$, where F is completely continuous, λ continuous, $0<m\leqq\lambda(x)\leqq M$. The set $f(S)$ is the object which here plays the rôle of cycle. The order $u(f, S, y_0)$ of a point y_0 with respect to $f(S)$ is defined by an approximation device which essentially reduces the situation to one of finite dimensionality, where known results apply. The order u is shown to remain constant if y_0 varies continuously or the cycle is deformed continuously, always with y_0 not in $f(S)$; the theorems of Rouché and Poincaré-Bohl follow. The order is the same for equivalent norms. The remaining results are proved only for strictly convex spaces. In this case, if f_1, f_2 are two mappings as above, then the two cycles are homotopic in $E-y_0$ if and only if $u(f_1, S, y_0)=u(f_2, S, y_0)$. If f maps S onto a like boundary S_1, the degree of the mapping is defined, and the analogues of two finite dimensional theorems are proved. *J. A. Clarkson.*

Taylor, A. E. The weak topologies of Banach spaces. Proc. Nat. Acad. Sci. U. S. A. **25**, 438–440 (1939). [MF 891]

Let $G\subset E^*$, where E^* is the conjugate space of a Banach space E. If $f_1, \cdots, f_n\in G$ and $\epsilon>0$, let $U(f_1, \cdots, f_n; \epsilon)$ denote the set of all points $x\in E$ for which $|f_i(x)|<\epsilon$, $i=1, 2, \cdots, n$. The family of all such sets, when taken as neighborhoods of the origin in E, gives rise to a linear topological space (l.t.s.) $\mathfrak{E}(G)$ whose points are the points of E. Let $\mathfrak{E}=\mathfrak{E}(E^*)$. Similarly, $G\subset E^{**}$ leads to the l.t.s. $\mathfrak{E}^*(G)$ and $\mathfrak{E}^*=\mathfrak{E}^*(E^{**})$. If G is the set of elements of E^{**} representable in the form $F(f)=f(x)$ for some $x\in E$ and all $f\in E^*$, the space formed with the elements of E^* is denoted by $\mathfrak{E}_*$. Various theorems are given which show that, if some one or more of the spaces E, E^*, $\mathfrak{E}$, $\mathfrak{E}^*$ and $\mathfrak{E}_*$ have one or more of certain properties (for example, reflexivity, separability, some form of completeness or compactness), then some other one of the spaces has some one of the properties. Finally, continuity properties of mappings from E_1 to E_2 are discussed in terms of corresponding l.t.s. *R. P. Agnew* (Ithaca, N. Y.).

v. Neumann, J. and Halperin, I. On the transitivity of perspective mappings. Ann. of Math. **41**, 87–93 (1940). [MF 1007]

This note gives a proof of the statement: If in a continuous geometry $a_1\sim a_2$, $a_2\sim a_3$, $\cdots$, $a_n\sim a_{n+1}$ and $a\cdot a_{n+1}=\Pi_{i=1}^{n+1}a_i$, then $a_1\sim a_{n+1}$. This proof differs from that given in the set of notes by J. v. Neumann [Continuous Geometry, Princeton, N. J., 1936] in a number of respects. For instance, the notions of independence and decomposition are considered modulo c and by transfinite induction it is shown that, if countable continuity on the lattice operations is replaced by Ω continuity for any cardinal Ω, the result still holds. *F. J. Murray* (New York, N. Y.).

ANALYSIS

***Warren, A. G. Mathematics Applied to Electrical Engineering.** Chapman & Hall, London, 1939. xv+384 pp. 15 s.

This is volume 9 of a series of monographs on electrical engineering edited by H. P. Young. A very large number of topics are treated with brevity in the manner of an engineering handbook. Numerous worked-out examples illustrate the text. Chapters 1–4 review complex numbers, differential and integral calculus. The treatment of limiting processes is in terms of sums of infinitesimals. An increment is said to approach a differential as a limiting value. Chapter 5 is on electrical physics and is entitled: A Brief Survey of Fundamental Electrostatic, Magnetic and Electrodynamic Relations. In chapter 6 the preceding mathematics and physics are combined in the solution of a number of fairly standard electrical problems. Chapter 7 contains an unusually clear introduction to the rotating plane vectors of electrical engineering. Inversion, vector loci and symmetrical components are also considered here. Chapters 8–12 take up selected topics from elementary differential equations and advanced calculus. Special emphasis is put on classical operational methods for solving linear constant-coefficient differential equations. More about simple differential equations and their applications in power and communication electrical engineering is contained in chapters 13 and 14. Elementary differential equations and their solution by series are the topics of chapters 15 and 16. The properties of Bessel Functions and sample applications to electrical problems are given in chapters 17 and 18. In chapter 19 some simple electrical partial differential equations are introduced. Chapter 20 is on the application of Fourier Series and harmonic analysis to wave forms which occur in electrical engineering. Chapter 21 is entitled: Heaviside's Operational Calculus. In it several of the methods used by Heaviside are compared with classical operational methods and are then applied to the solution of transmission line problems. The more recent

methods available in this field are not employed. The final chapter, 22, is on conformal representation and the use of conjugate functions in electrical problems.

Two appendices summarize the mathematical and electrical formulas. A third appendix is on dimensions. The volume concludes with a brief book bibliography.

J. L. Barnes (Merchantville, N. J.).

Theory of Sets, Theory of Functions of Real Variables

***Verblunsky, S. An Introduction to the Theory of Functions of a Real Variable.** Oxford University Press, Oxford, 1939. xi+169 pp. 12/6.

This book is based on the notes of lectures which the author delivered to students in their first year at the University of Manchester. The object of the author is to furnish students and teachers with a text which attains a higher level of correctness than is usual in books intended to be read by undergraduates.

Maximoff, Isaie. Sur les ensembles mesurables B dans l'espace transfini. Compositio Math. **7**, 201–213 (1939). [MF 794]

"L'espace transfini d'ordre i" consists of all sequences $[x_1, x_2, \cdots, x_\alpha, \cdots]$ of type ω_i whose elements are ordinal numbers less than ω_i. The elements x_α, for $\alpha \geqq \omega_0$, are essentially dummy indices and the topology is that of the product of countably many discrete spaces, each of cardinal number $\aleph_i$. The notion of Borel set is generalized by allowing $\aleph_i$ successive operations of union or intersection of $\aleph_i$ sets. A set belongs to the αth class $K_\alpha^i(\aleph_i)$ if it is the limit of a sequence of type ω_i of sets of lower classes. The initial class $K_0^i(\aleph_i)$ consists of these sets which are both open and closed. The notion of universal set of class is generalized. Universal sets of class 1 exist. Universal sets of all classes less than ω_{i+1} are asserted to exist. The proof of this last statement depends on the existence and continuity of certain functions [pp. 206–207]. The construction of these functions breaks down at the ω_0th step; their existence in the case $i=1$ would imply $2^{\aleph_0}=2^{\aleph_1}$. The existence proof is valid only for $\alpha<\omega_0$. *J. W. Tukey* (Princeton, N. J.).

Gutiérrez Novoa, L. The law of duality of sets of points. Bol. Mat. **12**, 198–201 (1939). (Spanish) [MF 456]

Calugaréano, Georges. Sur la suite des diamètres successifs d'un ensemble plan. C. R. Acad. Sci. Paris **209**, 409–411 (1939). [MF 375]

Essai d'une généralisation du diamètre transfini.

M. Brelot (Bordeaux).

Pauc, Chr. Considérations sur la surface représentative de la distance d'un point à un ensemble ponctuel plan. Rev. Sci. (Rev. Rose Illus.) **77**, 493–496 (1939). [MF 477]

Soient ξ_1, ξ_2, ξ_3 des coordinates rectangulaires dans un espace euclidien à trois dimensions; soit E un ensemble fermé dans un plan E d'équation $\xi_3=0$; soit $\rho(x)=\rho(\xi_1, \xi_2)$ la plus courte distance du point $x=(\xi_1, \xi_2, 0)$ dans E de E; et soit S la surface d'équation $\xi_3=\rho(\xi_1, \xi_2)$. Le point $x=(\xi_1, \xi_2, 0)$ n'appartienne pas à E, et le cercle de centre x et de rayon $\rho(x)$ n'ait en commun avec E qu'un seul point q ("point ordinaire"). L'auteur donne une démonstration géométrique du théorème de Golab [C. R. Acad. Sci. Paris **206**, 408 (1938)] que le paratingent de S en $(\xi_1, \xi_2, \rho(\xi_1, \xi_2))$ est alors le plan par ce point dont la trace sur E est la tangente de ce cercle en q. Une conséquence de ce théorème est une proposition de Bouligand [Géométrie infinitésimale directe, 1932, p. 105] sur la structure de l'ensemble de points ordinaires x obéissant à une équation $\rho(x)=\rho_0$. Le théorème lui même et sa démonstration peuvent être étendus à des ensembles fermés d'un espace euclidien quelconque. La fonction $\rho(x)$ admet en tout point p du complémentaire G de E en E et suivant toute direction θ une dérivée $\varphi(p, \theta)$ [De Misès, C. R. Acad. Sci. Paris **205**, 1354 (1937)]. L'auteur discute cette fonction $\varphi(p, \theta)$ définie en G la regardant comme un intégrant du calcul des variations. C'est un exemple d'un intégrant discontinu [mais semicontinu inférieurement] et non linéaire dont la valeur de l'intégrale le long de toute courbe rectifiable intérieure a son champ G ne dépend que des deux extrémités. Il est linéaire et continu dans les points ordinaires; dans les autres points il est discontinu et admet au plus une direction de linéarité.

P. Scherk (Watertown, Conn.).

Denjoy, A. Topological and metrical points of view in the theory of sets and functions of real variables. Duke Math. J. **5**, 806–813 (1939). [MF 821]

The author gives a general discussion of methods to be used in studying such questions as may for example occur at a boundary of a region of regularity and distinguishes between methods which are topological and those which are metrical. He outlines the metrical and topological approach to problems of this type. *N. Levinson.*

Cassina, U. Curva di Peano in base due. Period. Mat. **19**, 113–125 (1939). [MF 561]

Lyche, R. Tambs. Une expression simple pour une fonction continue sans dérivée. Norske Vid. Selsk. Forh. **12**, 45–48 (1939). [MF 868]

Let $g(x)$ denote the distance of the real number x from the nearest integer. It is known that $f(x)=\sum_{n=0}^{\infty}2^{-n}g(2^n x)$ is a continuous function having nowhere a derivative [van der Waerden, Math. Z. **32**, 474 (1930); T. H. Hildebrandt, Amer Math. Monthly **40**, 547 (1933)]. It is shown that if (1) $x=2^{-\alpha_1}+2^{-\alpha_2}+\cdots$ $(1\leqq\alpha_1<\alpha_2<\cdots)$ is the binary expansion of a real number x $(0<x\leqq 1)$, then

$$(2)\quad f(x)=\frac{\alpha_1}{2^{\alpha_1}}+\frac{\alpha_2-2}{2^{\alpha_2}}+\frac{\alpha_3-4}{2^{\alpha_3}}+\cdots+\frac{\alpha_i-2(i-1)}{2^{\alpha_i}}+\cdots.$$

In case (1) is terminating, then (2) is supposed to terminate accordingly. The author says: "En choisissant (2) pour définition de la fonction $f(x)$ les propriétés essentielles de cette fonction découlent d'une manière très simple de la définition." *I. J. Schoenberg* (Waterville, Me.).

Lyche, R. Tambs. Une courbe simple sans courbure. Norske Vid. Selsk. Forh. **12**, 49–52 (1939). [MF 1028]

The continuous nowhere differentiable function $f(x)$ of the foregoing review is being integrated and an explicit expression of its indefinite integral given.

I. J. Schoenberg (Waterville, Me.).

Popoff, Kyrille. Sur une extension de la notion de dérivée. Monatsh. Math. Phys. **48**, 103–120 (1939). [MF 630]

For $y=f(x)$ the author considers $P[x, f(x)]$, the family

of lines through P and the distance D from $Q[x+u, f(x+u)]$ to each line. If $\alpha=\alpha(u)$ is the inclination of the line for the extreme values of

$$(1)\qquad \int_0^u D^3 du, \qquad (2)\qquad \int_0^u D\varphi(u)du,$$

φ a weight function, $\lim \tan\alpha$ as $u\to 0$ is related to the slope of the graph. In (1) $\tan\alpha$ is two valued and, if $f'(x)$ exists, the limits of $\tan\alpha$ are the slope of the tangent and normal at P. In (2) the limit is $f'(x)$ if the derivative exists. Applying (1) to space curves and surfaces certain formulas of differential geometry are generalized. In (2) the classical formulas for the derivatives of sums, products, quotients and composite functions hold under broad conditions. For $\varphi(u)=u^{-1}$, the classical difference quotient is replaced by its integral mean. If $D_n(h)$ is the nth integral mean of $[f(x+h)-f(x)]/h$, bounded in h, and $D(h)=\lim D_n(h)$ for $0<h\leqq h_0$, then $\lim D(h)$ exists for $h\to+0$. For $\varphi(u)\equiv 1$, $\lim \tan\alpha$ exists if $|f(x|h)|=|f(x+h)-f(x)|<M|h|$ and

$$f(x|h)-ah^{-1}\int_0^h \left(\frac{u}{h}\right)^{s-2} f(x|u)du$$

does not change sign for some $a>0$ [C. R. Acad. Sci. Paris **207**, 111–112 (1938)]. *L. W. Cohen* (Lexington, Ky.).

Anghelutza, Th. Sur une propriété des polynômes. Bull. Sci. Math. **63**, 239–246 (1939). [MF 738]

It is shown that if $f(x)$ is continuous on (a, b) and

$$\lim_{h\to 0}\frac{1}{h^p}\sum_{j=0}^{p}\binom{p}{j}f(x+(p-j)h)=0,$$

uniformly in x, then $f(x)$ is a polynomial of degree $p-1$. The necessity of the uniformity condition is indicated by the example $f(x)=|x|$ and $p=2$. That the result of Schwarz to the effect that the vanishing of the generalized second derivative of $f(x)$ implies that $f(x)$ is linear does not generalize is shown by the example, due to Montel: if $f(x)=x|x|$ then

$$\lim_{h\to 0}[f(x+2h)-4f(x+h)+6f(x)-4f(x-h)+f(x-2h)]/h^4=0.$$

L. W. Cohen (Lexington, Ky.).

Jessen, Børge. Abstract theory of measure and integration. 4. Mat. Tidsskr. B **1939**, 7–21 (1939). (Danish) [MF 718]

This is a continuation of an elegant expository survey of the whole theory. The fourth part contains the theory of measure and integration in product spaces, including products of enumerably many spaces. The first three papers appeared in the same journal in 1934, 1935 and 1938, respectively. *W. Feller* (Providence, R. I.).

Ridder, J. Sur la totalisation par rapport à une fonction à variation bornée généralisée. C. R. Acad. Sci. Paris **209**, 623–625 (1939). [MF 707]

The function $\alpha(x)$ is said to be of generalized bounded variation (VBG) on (a, b) when (a, b) is the sum of a denumerable set and a denumerable sequence of perfect sets P_j on each of which $\alpha(x)$ is of bounded variation. $F(x)$ is then generalized absolutely continuous (ACG) with respect to $\alpha(x)$ if it is absolutely continuous on each P_j with respect to the function

$$U_j=V_{P_j}(x)+\sum_a^x\{b_n^j-a_n^j\}+\{x-a_n^j(x)\},$$

where V_{P_j} is the total variation of a function equal to $\alpha(x)$ on P_j and linear in the intervals (a_n^j, b_n^j) contiguous to P_j. $F(x)$ satisfies the condition $N_{\alpha(x)}$ on (a, b) if, for every $E_j\subset P_j$ with the measure of E_j with respect to V_{P_j} equal to zero, $mF(E_j)=0$. It is shown that all functions (ACG) with respect to $\alpha(x)$ satisfy the condition $N_{\alpha(x)}$.

The author then defines the inferior derivative of the first kind on the right at a point ξ of a set E as the lower limit of $\{F(x)-F(\xi)\}/\{\alpha(x)-\alpha(\xi)\}$, $\xi\varepsilon E$, $x\varepsilon E$, $x\to\xi$, $\alpha(x)>\alpha(\xi)$ and of the second kind by changing $\alpha(x)>\alpha(\xi)$ to $\alpha(x)<\alpha(\xi)$, and similarly for derivatives on the left. It is then shown that, if $\alpha(x)$ is of bounded variation on a perfect set P, E a set contained in P with (1) the extreme right hand derivatives over P less than M in numerical value on E, (2) for all points ξ of E which are points of accumulation of P, x sufficiently close to ξ and $\alpha(x)=\alpha(\xi)$ implies $F(x)=F(\xi)$, then $\mathfrak{m}F(E)\leqq M\mathfrak{m}V_P(E)$, $\mathfrak{m}$ denoting exterior measure. It is also shown that, if $\alpha(x)$ and $F(x)$ are (VBG) on (a, b) and, at points of discontinuity of $\alpha(x)$, $\{F(x)-F(\xi)\}/\{\alpha(x)-\alpha(\xi)\}$ tends to a finite limit $A(\xi)$ as $x\to\xi$, then a necessary and sufficient condition that $F(x)$ be (ACG) is that it possess property $N_{\alpha(x)}$. *R. L. Jeffery.*

Ridder, J. Nouvelles propriétés de la totalisation par rapport à une fonction à variation bornée généralisée. C. R. Acad. Sci. Paris **209**, 670–672 (1939). [MF 712]

This is a continuation of the paper reviewed above. The function $\alpha(x)$ is (VBG) on (a, b), and P_j are the perfect sets over which $\alpha(x)$ is of bounded variation. $F(x)$ is such that in each point ξ of P_j, except perhaps for a denumerable set, two extreme derivatives over P_j, the one superior, the other inferior, of different kinds but on the same side, are finite; while for x on this side and sufficiently close to ξ, $\alpha(x)=\alpha(\xi)$ implies $F(x)=F(\xi)$; then $F(x)$ satisfies the condition $N_{\alpha(x)}$ on (a, b). If, in addition, at the points ξ of discontinuity of $\alpha(x)$, $\{F(x)-F(\xi)\}/\{\alpha(x)-\alpha(\xi)\}$ tends to a finite limit $A(\xi)$ which is bounded, then $F(x)-F(a)$ is an indefinite integral of a function $f(x)$ with respect to $\alpha(x)$ in the sense of the author [Math. Ann. **116**, 98 (1938)], where $f(x)=A(x)$ in the points of discontinuity of $\alpha(x)$, and $f(x)$ equals one of the two derivatives introduced above, where this derivative exists, and $f(x)$ is arbitrary elsewhere. In the special case of one perfect set P_1 which coincides with (a, b) there is obtained a solution to the problem proposed by Lebesgue, Leçons sur l'integration, 2nd ed., 1928, p. 307.

R. L. Jeffery (Wolfville, N. S.).

Bochner, S. Additive set functions on groups. Ann. of Math. **40**, 769–799 (1939). [MF 322]

A linear mean value function $M_x f(x)$ defined on a linear class C of bounded functions $f(x)$ on a set $\mathfrak{G}$, if continuous on C in the l.u.b. metric, is a Riemann integral with respect to some Jordan volume v defined on a Jordan field $\mathfrak{F}_C$ of sets in $\mathfrak{G}$. The class C is dense in the space R_p ($p\geqq 1$) of all Riemann integrable functions under the metric

$$\left(\int_{\mathfrak{G}}|f(x)-g(x)|^p dv\right)^{1/p}.$$

The functional $M_x f(x)$ can be extended to a Lebesgue integral if and only if $M_x f_n(x)\to 0$ whenever $f_n(x)\to 0$, $|f_n(x)|\leqq K$. The mean value

$$M_x f(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} f(t)dt$$

of continuous almost periodic functions is one that cannot be extended. The Nikodym theorem on representation of a.c. set functions is proved for the case of Jordan volume. It takes the following form in R_1. In the space V_1 of additive set functions $F(E)$ which vanish with $v(E)$ and with $\|F\|=\text{l.u.b.}\sum_i|F(E_i)|$ (the l.u.b. taken over partitions of $\mathfrak{G}$) the integrals of step functions are dense. If $\mathfrak{G}$ is a group and (i) C is group invariant, that is, $f(x^{-1})$, $f(ax)$, $f(xa)\varepsilon C$ when $f(x)\varepsilon C$, (ii) the generated Jordan field is group invariant and (iii) the volume is both right and left invariant, then an a.c. set function $F\varepsilon V_1$ vanishes identically if $\int f(x)dF=0$ for all $f\varepsilon C$. The Fourier expansion $F(E)\sim\sum_\alpha c_\alpha\phi_\alpha(x)$ with respect to a not necessarily countable set of orthonormal functions ϕ_α may have an uncountable number of its coefficients $c_\alpha\neq 0$ but this is no longer true if F is a.c. A uniqueness theorem for right invariant $F(E)$ is given. A function $F\varepsilon V_C$, the space conjugate to a group invariant C, is a.c. provided the function $T(a)=F(Ea^{-1})$ on $\mathfrak{G}$ is almost periodic. This is a partial generalization of a theorem of Plessner which states that a measurable $F(x)$ is a.c. if $\int|d_x(F(x+h)-F(x))|\to 0$ with h. In the case where C is the set of Bohr functions on the line $-\infty<x<\infty$ some general conditions are given which are necessary and sufficient for the series $\sum_\alpha a_\alpha e^{i\alpha x}$ to be the Fourier expansion of a function $F\varepsilon V_1$. *N. Dunford.*

Aronszajn, N. Quelques recherches sur l'intégrale de Weierstrass. I. Rev. Sci. (Rev. Rose Illus.) **77**, 490–493 (1939). [MF 460]

The author considers a space with a distance $r(p, q)$ subject only to the condition that r is real-valued and $r(p, p)=0$, and a curve $p=p(t)$, $0\leqq t\leqq 1$. This determines a distance $r(t', t'')$ on the interval $(0, 1)$, in terms of which a number of related functions are defined. Among these are the sums over partitions of the interval (α, β), $\lambda_r\{t_i\}=\sum_r(t_i, t_{i+1})$, the "upper and lower integrals of Weierstrass along the curve $p=p(t)$," namely, $\bar{\lambda}_r(\alpha, \beta)=\limsup\lambda_r\{t_i\}$, $\underline{\lambda}_r(\alpha, \beta)=\liminf\lambda_r\{t_i\}$, and the upper and lower bounds $\bar{I}_r(\alpha, \beta)=\bar{B}\lambda_r\{t_i\}$, $\underline{I}_r(\alpha, \beta)=\underline{B}\lambda_r\{t_i\}$. Relations between these functions and their possible properties, such as continuity, additivity, and so on, form the main topic of the paper.

L. M. Graves (Chicago, Ill.).

Theory of Functions of a Complex Variable

von Koppenfels, Werner. Konforme Abbildung besonderer Kreisbogenvierecke. J. Reine Angew. Math. **181**, 83–124 (1939). [MF 590]

For regions bounded by arcs of circles, the author investigates the problem of determining explicit mapping functions analogous to the Schwarz-Christoffel formula for maps of a half-plane on regions bounded by polygons. The mapping function is the quotient of two independent solutions of a linear differential equation of the second order. This equation reduces, for the case of three circular sides, to the hypergeometric differential equation, so that the problem is completely solvable for three sides. For four circular sides, the author is able to proceed in the special cases when the angles are odd multiples of $\pi/2$ ($\pi/2$, $\pi/2$, $\pi/2$, $3\pi/2$ and $\pi/2$, $\pi/2$, $3\pi/2$, $3\pi/2$), since then the fact that the opposite sides are perpendicular allows the boundary to be imbedded in a Steiner network of circles. In these special cases the problem of determining the dependence of the parameters of the differential equation on the geometric constants of the curvilinear quadrangle is solved. Application is made to the Hermite-Halphen theory of the Lamé differential equation. *E. F. Beckenbach* (Houston, Tex.).

Walsh, J. L. On the circles of curvature of the images of circles under a conformal map. Amer. Math. Monthly **46**, 472–485 (1939). [MF 359]

Let $w=f(z)$ map the circle $|z|<1$ conformally onto a simple region R in the extended w-plane. The paper studies the circles of curvature T_ρ of the images of an arbitrary circle of radius ρ under both transformations, $w=f(z)$ and its inverse, especially with reference to intersections of the circles T_ρ with the boundaries in the respective planes. Let $f(z)=z+a_2z^2+\cdots$ near $z=0$, and let C_ρ be a circle in the z-plane through $z=0$, T_ρ the circle of curvature of its image at $w=0$. After giving a characterization of the set of the T_ρ for all circles C_ρ of given radius ρ, the author proves a number of results of which the following are examples: For every region R there exist two numbers ρ_1, ρ_2, $\frac{1}{4}\leqq\rho_1\leqq\rho_2\leqq\infty$ such that for $\rho_2\leqq\rho\leqq\infty$ each T_ρ cuts the boundary B of R; for $\rho_1\leqq\rho<\rho_2$, a given T_ρ need not cut B, but for each ρ some T_ρ cuts B; for $\rho<\rho_1$ no T_ρ cuts B. Every T_ρ cuts the circle $|w|=r$ if $r\leqq 2\rho[1+2|a_2|\rho]^{-1}$ and no T_ρ cuts $|w|=r$ if $r>2\rho[1-2|a_2|\rho]^{-1}>0$. These results are extended to circles C_ρ not passing through $z=0$. An interesting theorem in this connection is that no circle of curvature of the image of a circle $|z|=r<2-\sqrt{3}$ can cut B. Similar theorems are obtained for the images of circles in the w-plane under the inverse transformation. Finally, some of the results are expressed in general form in terms of non-euclidean geometry.

S. E. Warschawski (St. Louis, Mo.).

Courant, R. Conformal mapping of multiply connected domains. Duke Math. J. **5**, 814–823 (1939). [MF 822]

This paper is a contribution to the theory of conformal mapping treated by methods developed in the problem of Plateau. Given, in the xy-plane, a domain G bounded by k Jordan curves, there are various types of normal domains upon which G can be mapped conformally. The author is concerned with normal domains located upon Riemann surfaces over the uv-plane and defined as follows. Take, over the uv-plane, k unit circles and connect them in branch-points of total multiplicity $2k-2$, then affix to this figure $p\geqq 0$ full planes with two branch-points each. The author proves that for each $p\geqq 0$ the domain G can be mapped conformally upon a domain of the type just described. The method of proof is a simplification of that used by the author previously [see Ann. of Math. **38**, 679 ff. (1937)]. The simplification is mainly due to the fact that the author uses, instead of the Dirichlet integral, the integral of $(x_u-y_v)^2+(x_v+y_u)^2$ which was considered by Riemann in his thesis.

T. Radó (Columbus, Ohio).

de Possel, René. Sur la représentation conforme d'un domaine à connexion infinie sur un domaine à fentes parallèles. J. Math. Pures Appl. **18**, 285–290 (1939). [MF 807]

Let D be a single sheeted domain in the z-plane, of finite or infinite connectivity and containing the point at infinity. Then D can be mapped conformally on a domain $\bar{D}$ in the w-plane such that the frontier elements of $\bar{D}$ consist exclusively of points and of segments of lines parallel to the real axis, and such that in the neighborhood of infinity the mapping function has a representation of the form $w=z+a/z+\cdots$. The author's present proof of this known

result is an improvement of his former proof [Nachr. Ges. Wiss. Göttingen **1931**, 199–202] in that it is independent of the special case of the conformal mapping of a simply connected domain. The method consists of establishing the existence of a mapping function for D which satisfies the above normalization at infinity and maximizes the real part of a, and then showing that this function gives a map $\bar{D}$ of the desired sort. *E. F. Beckenbach* (Houston, Tex.).

Jørgensen, Vilhelm. On a corollary to Picard-Landau's theorem. Mat. Tidsskr. B **1939**, 1–6 (1939). (Danish) [MF 717]

By Picard-Landau's theorem there is a $\phi(\alpha)$ such that, if $g(z)=\alpha+\beta z+\cdots$ is regular in $|z|<1$ and not equal to 0 or 1, then $|\beta|\leqq\phi(\alpha)$. The author now proves: (i) Let $f(z)$ be meromorphic in $|z|<1$, $f(z)=z^{-n}+a_{n-1}z^{-n+1}+\cdots$ for small $|z|$, and $f(z)\neq a, b, c$. Then

$$(*)\qquad |c-b|\leqq\left|\frac{c-a}{b-a}\right|\phi\left(\frac{c-b}{c-a}\right).$$

Conversely, to any three numbers a, b, c satisfying (*), there corresponds a $f(z)$ of the described sort. (ii) If $g(z)=z^n+a_{n+1}z^{n+1}+\cdots$ is regular for $|z|<1$ and not equal to $1/(ae^{i\alpha})$, with c real and $a\geqq 16$, then $g(z)$ takes on all values of the circular ring

$$\frac{40}{ae^{c/2}}<\left|w-\frac{1}{ae^{2c}}\right|<\frac{e^{c/2}}{40a}.$$

W. Feller (Providence, R. I.).

Shah, S. M. On integral functions of perfectly regular growth. J. London Math. Soc. **14**, 293–302 (1939). [MF 436]

L'auteur étudie la relation entre le nombre $n(r)$ des zéros d'une fonction entière $f(z)$ d'ordre ρ positif non entier et la croissance de $\log M(r)$ [$M(r)$ maximum de $|f(re^{i\phi})|$] dans le cas des croissances parfaitement régulières. Il montre que si l'on a

$$0<\lim_{r=\infty}\frac{\log M(r)}{r^\rho}=C<\infty,$$

le rapport $n(r)/r^\rho$ peut ne pas avoir de limite; et que si

$$(1)\qquad 0<\lim_{r=\infty}\frac{n(r)}{r^\rho}=C'<\infty,$$

le rapport de $\log M(r)$ à r^ρ peut ne pas avoir de limite. La première propriété résulte de l'étude des zéros de la somme $f(z)+g(z)$, où $f(z)$ a ses zéros alignés tandis que $g(z)$ est donné par une série lacunaire. La seconde proposition découle en réalité du fait que, moyennant (1), $\log M(r)$ est défini asymptotiquement par les zéros dont le module est compris entre $k(r)r$ et $r/k(r)$, $k(r)$ tendant lentement vers l'infini, ce qui permet de faire osciller $\log M(r)/r^\rho$ entre les limites différentes qu'il aurait pour deux choix convenables des arguments. Les résultats de l'auteur s'étendraient de suite au cas où la constante ρ serait remplacée par un ordre précisé L non entier. *G. Valiron* (Paris).

Robinson, Raphael M. On numerical bounds in Schottky's theorem. Bull. Amer. Math. Soc. **45**, 907–910 (1939). [MF 780]

Gemäss dem Schottky'schen Satz besitzt eine in $|z|<r$ reguläre, von -1 und $+1$ verschiedene Funktion $f(z)$ für ihren Betrag in $|z|\leqq r<1$ eine Schranke $K(r, a)$, die nur von r und $f(0)=a$ abhängt. Der Verfasser beweist die asymptotische Formel

$$(1)\qquad 8K(r, a)\sim(8|a|)^{(1+r)/(1-r)},\qquad a\to\infty,\ r \text{ fest}.$$

Damit verallgemeinert er ein Resultat von A. Ostrowski [Comment. Math. Helv. **5**, 55–87 (1933)] mit 16 an Stelle von 8. (1) ergibt sich unmittelbar aus den beiden Abschätzungen

$$(2)\qquad 8K(r, a)-10<(8|a|+10)^{(1+r)/(1-r)}$$

und

$$(3)\qquad 8K(r, a)+10>(8|a|-10)^{(1+r)/(1-r)},$$

sofern $8|a|-10>0$. (2) und (3) verschärfen wesentlich die numerischen Schranken von L. Ahlfors [Trans. Amer. Math. Soc. **43**, 359–364 (1938)] und vom Referenten [Comment. Math. Helv. **7**, 159–170 (1935)]. Der Beweis gelingt den Verfasser mit Hilfe elliptischer Modulfunktionen.

A. Pfluger (Freiburg).

Nehari, Zeev. Sur la déformation de la frontière par les fonctions univalentes convexes. C. R. Acad. Sci. Paris **209**, 781–783 (1939). [MF 864]

Let

$$f(z)=z+a_2z^2+\cdots$$

be an analytic function, holomorphic and univalent for $|z|\leqq 1$, which represents the unit circle on a convex domain. Then for $|\phi_1-\phi_2|<\pi$ the author establishes the inequality

$$|f(e^{i\phi_1})-f(e^{i\phi_2})|\geqq\tan(|\phi_1-\phi_2|/4).$$

The value $\tan(|\phi_1-\phi_2|/4)$ cannot be replaced by a larger value. The proof is based upon the observation that the function

$$f(ze^{i\psi})-f(ze^{-i\psi}),\qquad 0<\psi<\pi,$$

is univalent and star-like with respect to the origin for $|z|<1$ whenever $f(z)$ is holomorphic, univalent and convex for $|z|<1$.

Throughout the proof some misprints have resulted in the senses of several inequalities becoming reversed.

M. S. Robertson (New Brunswick, N. J.).

Maitland, B. J. A note on functions regular and bounded in the unit circle and small at a set of points near the circumference of the circle. Proc. Cambridge Philos. Soc. **35**, 382–388 (1939). [MF 543]

Suppose that $f(z)$ is regular and $|f(z)|<1$ in the unit circle. Suppose also that $|f(z)|<\mu<1$ at a set of points ξ in the annulus $1-\delta<|z|<1$, where $0<\delta<\frac{1}{2}$. Then, if ξ^* is the radial projection of ξ onto $|z|=1$,

$$|f(re^{i\theta})|<\mu^{K(1-\delta)(1-r)(m/2\pi)},$$

where K is a numerical constant, certainly not less than $1/30$, and m is the measure of ξ^*. There is a set $\xi_1\subset\xi$ consisting of a finite number of circular arcs concentric with $|z|=1$; and the radial projections on $|z|=1$ of these arcs are non-overlapping and form a set of measure greater than $m-\epsilon$, $\epsilon>0$. The harmonic function

$$U(z, \xi_1)=\frac{1}{\pi}\int_{\xi_1}\frac{|x|}{1-|x|^2}\log\left|\frac{1-z\bar{x}}{z-x}\right|d(\arg x)$$

vanishes on $|z|=1$, and cannot be greater than k in the neighborhood of ξ_1. Here k is a numerical constant. An application of the maximum principle shows that

$$\log|f(z)|\leqq\frac{1}{k}U(z, \xi_1)\log\mu$$

inside the unit circle. An estimate is obtained for the lower bound of U as a function of $|z|$, and the result follows.
A. C. Schaeffer (Palo Alto, Calif.).

Dinghas, Alexander. Zur Werteverteilung einer Klasse transzendenter Funktionen. Math. Z. **45**, 507–510 (1939). [MF 405]

Let $w(z)$ be a transcendental integral function of order $\rho<\frac{1}{2}$ and let a be a (finite) complex number. Let $n(r, a)$ denote the number of zeros of $w(z)-a$ in $|z|\leqq r$, and $S(r, w)$ Ahlfors' characteristic function [Acta Math. **65**, 157–194], and let

$$L_r=\int_0^{2\pi}\frac{|w'|}{1+|w|^2}|dz|.$$

The author proves the following relation (which is analogous to a theorem of Ahlfors [loc. cit.] on a certain class of meromorphic functions): $S(r, w)=n(r, a)+O(L_r)$ as $r\to\infty$, $r\in E$, where E is a certain set whose lower logarithmic density is positive. *S. E. Warschawski* (St. Louis, Mo.).

Pfluger, A. Die Wertverteilung und das Verhalten von Betrag und Argument einer speziellen Klasse analytischer Funktionen. II. Comment. Math. Helv. **12**, 25–65 (1939). [MF 498]

In section 1 of this paper [Comment. Math. Helv. **11**, 180–213 (1938–39): to be cited as (I)] the author introduced the class of entire functions of finite order whose zeros form sets "measurable" with respect to a Lindelöf proximate order $\rho(r)$ [I, 204]. He showed in particular [I, 206] that, if $f(z)$ is such a function, of non-integral order ρ, having the Phragmén-Lindelöf indicator $h(\varphi)$ (with respect to $\rho(r)$), then, for $-\pi\leqq\varphi\leqq\pi$,

$$\lim_{r\to\infty}{}^{*}\frac{1}{V(r)}\log|f(re^{i\varphi})|=h(\varphi), \tag{1}$$

where $V(r)\sim r^{\rho(r)}$ and $\lim^{*}$ means that the limit is taken as $\rho\to\infty$ through a certain set of linear density unity. In section 3 he now defines a function, analytic in an angle $\alpha<\varphi<\beta$ and of positive order ρ and proximate order $\rho(r)$ there, to be of regular asymptotic growth in the angle if (1) holds for $\alpha<\varphi<\beta$. He shows that (qualitatively speaking) such a function is also regular with respect to the distribution of the values of its argument, of its zeros and (when $h(\varphi)>0$) of its a-points. The wealth of detailed quantitative information which the author obtains cannot be adequately summarized in a brief review, but some of the most general results will be quoted. If $f(z)$ is of regular asymptotic growth, its zeros form a set measurable with respect to $\rho(r)$, and the measure function of the zeros has a simple expression in terms of $h(\varphi)$ [converse of the result cited from (I)]. The relation

$$\lim_{r\to\infty}\frac{1}{V(r)}\int_{\alpha'}^{\beta'}d\{\arg f(re^{i\varphi})\}=\rho\int_{\alpha'}^{\beta'}h(\varphi)d\varphi \tag{2}$$

holds if $\alpha<\alpha'<\beta'<\beta$; conversely, if $f(z)$ is analytic in $\alpha'\leqq\alpha\leqq\beta'$ and (2) holds, $f(z)$ is of regular asymptotic growth in the open angle. Results on the distribution of the values of $f(z)$, and in particular of its Borel directions, are deduced. For example, the author proves and generalizes a theorem of M. L. Cartwright [Proc. London Math. Soc. (2) **38**, 440 (1935)] concerning Borel directions of a class of entire functions (this class turns out to be precisely the class of entire functions of regular asymptotic growth).

In section 2, the author investigates the properties of the envelope of the family of lines

$$x\cos\rho\varphi+y\sin\rho\varphi-h(\varphi)=0,\qquad \alpha<\varphi<\beta,$$

where $h(\varphi)$ is a function having the properties possessed by the indicator of some function analytic in the angle $\alpha<\varphi<\beta$. The envelope may have corners and both convex and concave arcs, but has no inflection points; $h(\varphi)$ and the "relative length" of the envelope (convex and concave arcs being measured with opposite signs) are expressible in terms of each other. The results furnish elegant geometric interpretations of such relations as (2) and the relation between the measure function of the zeros and the indicator.
R. P. Boas, Jr. (Durham, N. C.).

Whittaker, J. M. and Wilson, R. Fabry's theorem and the isolated essential singularity of finite exponential order. J. London Math. Soc. **14**, 202–208 (1939). [MF 244]

A classical theorem of Fabry asserts that, if $f(z)=\sum_{r=0}^{\infty}c_rz^r$ in the unit circle $|z|<1$ with $\lim_{n\to\infty}c_{n+1}/c_n=1$, then the point $z=1$ is a singular point of $f(z)$. The authors of this paper give a converse of Fabry's theorem. We give two definitions. If a function $f(z)$ regular in $|z|<1$ possesses the property that the function $\varphi(z)=f(z/(1+z))$ is an integral function of finite order, then $f(z)$ is regular everywhere on the circle $|z|=1$ with the exception of the point $z=1$, where it is said to possess an essential singularity of finite exponential order. A subsequence $\{n'\}$ of $\{n\}$, where n is the sequence of positive integers, is said to be of density 1 if $\lim_{n\to\infty}N(n)/n=1$, $N(n)$ denoting the number of numbers of the subsequence $\{n'\}$ not greater than n. The authors prove as their main result the theorem: If $f(z)=\sum_{r=0}^{\infty}c_rz^r$, regular in $|z|<1$, has on the circle $|z|=1$ only one singular point which is an isolated essential singularity of finite exponential order at $z=1$, then $\lim_{n'\to\infty}c_{n'}/c_{n'+1}=1$, where $\{n'\}$ is some subsequence of $\{n\}$ of density 1. An interesting feature of the proof is an extension of a well-known theorem of Wigert. *W. Seidel* (Rochester, N. Y.).

Biggeri, Carlos. A theorem on the singularities of Dirichlet series. Bol. Mat. **12**, 255–256 (1939). (Spanish) [MF 897]

Biggeri, Carlos. Sur les singularités des fonctions analytiques définies par des séries de Dirichlet. C. R. Acad. Sci. Paris **209**, 979–980 (1939). [MF 1198]

Proof of the following theorem: If the arguments φ_n of the coefficients of a Dirichlet series of type λ_n are such that (i) $\cos\varphi_n\geqq 0$, (ii) $\lim[\cos\varphi_n]^{1/\lambda_n}=1$, then the real point of the line of convergence of the series is a singular point of the function represented by the series [cf. Boll. Un. Mat. Ital. **15**, 209, 214; **16**, 82, 178]. *F. Bohnenblust.*

von Stachó, Tibor. Randsingularität Laplacescher Transformierten. Monatsh. Math. Phys. **48**, 408–418 (1939). [MF 657]

The author proves that if σ is the abscissa of convergence of the Laplace integral

$$\int_0^{\infty}\tau(t)e^{i\alpha(t)}e^{-st}dt=f(s),$$

then $s=\sigma$ is a singular point of $f(s)$ provided (1) $\tau(t)$ is integrable over every finite range, (2) $\alpha(t)$ is continuous, and (3) $\alpha'(t)\log t\to 0$ as $t\to\infty$. The author raises the ques-

tion whether the theorem is true under the weaker assumption $\alpha'(t)\to 0$. The proof is a clever adaptation of classical methods in gap theory. *E. Hille* (New Haven, Conn.).

Heins, Maurice H. Extremal problems for functions analytic and single-valued in a doubly-connected region. Amer. J. Math. **62**, 91–106 (1940). [MF 960]

The following theorem is known as the principle of the harmonic majorant: Let $f(z)$ be analytic in a region G with $|f(z)|$ single valued, and let $u(z)$ be a single valued harmonic function in G with the property that

$$\overline{\lim}\,(\log|f(z)|-u(z))\leqq 0$$

at every point of the boundary of G. Then $\log|f(z)|\leqq u(z)$ in G, and the equality holds only when $f(z)\equiv e^{u+iv}$. Since e^{u+iv} is in general not single valued if G is multiply connected, we can use the strict inequality $\log|f(z)|<u(z)$ when we restrict $f(z)$ to the class of single valued functions and G to multiply connected regions, and can replace $u(z)$ by a somewhat smaller function. In case e^{u+iv} is not single valued, the author finds the exact bound for $\log|f(z)|$ and the associated extremal functions in a doubly connected region. By consideration of functions of form $\psi(z)=f(z)/e^{u+iv}$, the problem is reduced to finding l.u.b. $|\psi(z)|$ in G for functions $\psi(z)$ analytic in G and such that $|\psi|<1$ in G and $\psi(z)$ becomes $e^{-i\theta}\psi(z)$, $\theta\neq 0$, when z describes a single closed circuit around one of the continua forming the boundary of G. By introducing the Poincaré uniformization function, the problem is further reduced to finding l.u.b. $|w(0)|$, where $w(z)$ is analytic and $|w|<1$ in $|z|<1$, and $w(T(z))=e^{-i\theta}w(z)$, where T is the hyperbolic transformation with respect to which the uniformization function is automorphic. This last problem is solved by means of an application of the Pick-Nevanlinna interpolation theory to the solution of the functional equation $w(T(z))=U(w(z))$, where U and T are given non-euclidean transformations of the unit circle. The author also solves by a similar method an extremal problem in connection with the principle of hyperbolic measure. Let $w=w(z)$ be a single valued analytic function in G_z assuming values in G_w, where G_z and G_w are of hyperbolic type. Then the ratio $d\sigma_w/d\sigma_z\leqq 1$, where $d\sigma_w$ and $d\sigma_z$ are the elements of hyperbolic measure in G_w and G_z, respectively. If G_z is multiply connected, the inequality is in general strong. The author finds the exact bound for $d\sigma_w/d\sigma_z$ and the functions attaining that bound in the case G_z is doubly connected. The problems of the paper are similar in nature to those of other recent investigators, notably Teichmüller, Carlson, Aumann, and Carathéodory. The author's method is a very general one, and enables him to solve a large class of problems, including some treated by the above mentioned authors.

J. W. Green (Rochester, N. Y.).

Brelot, M. Quelques applications aux fonctions holomorphes de la théorie moderne du potentiel et du problème de Dirichlet. Bull. Soc. Roy. Sci. Liége **1939**, 385–391 (1939). [MF 833]

If $f(z)$ is an analytic function of the complex variable z in a domain Ω, then $|f(z)|$ and $\log|f(z)|$ are subharmonic in Ω. As a consequence, the theory of subharmonic functions admits of important applications in the theory of analytic functions of a complex variable [see, for example, Radó, Subharmonic functions, Ergebnisse der Mathematik, vol. 5, 1933]. The purpose of the present paper is to give examples of new applications to refinements of the principle of maximum concerned with inferences from limitations upon the absolute value of an analytic function in the vicinity of a portion of the boundary of its domain of definition. Far-reaching extensions of recent results of Privaloff, F. Riesz, R. Nevanlinna and others are obtained. In contradistinction to earlier applications, the ones discussed by the author imply an unrestricted use of the deepest and most powerful results of recent research in subharmonic functions and in the Dirichlet problem. These applications suggest, in the opinion of the author and also of the reviewer, that the treatment of the theory of analytic functions of a complex variable by methods which are independent of potential theory cannot be insisted upon if the ultimate in elegance and insight is to be achieved. *T. Radó.*

Koebe, Paul. Iterationstheorie der hyperbolischen Uniformisierungsgrössen vom Geschlecht Null. Ber. Verh. Sächs. Akad. Wiss. Leipzig **91**, 135–192 (1939). [MF 668]

This paper is a direct continuation of a previous paper: Iterationstheorie der niederen Uniformisierungsgrössen [same journal (1937)]; likewise, it refers to older publications of the author, amplifying results and providing existence and convergence proofs without use of normal families of functions. The objective of the present paper is the conformal mapping by linear polymorphic functions of Riemann surfaces over the simple z-plane with fixed positions and fixed multiplicities of the branchpoints on the upper half plane, or on the whole plane except point sets on the real axis. The mapping is effected by a convergent process of iterated "fundamental transformations," which means transformations of a Riemann surface with two branchpoints of a finite or infinite order into a simple plane. Sequences of such transformations are used to smooth out the branchpoints successively and thus to attain the simple image plane as a limit.

The simplest type considered, generalizing the case of the elliptic modular function, is a Riemann surface with $n\geqq 3$ real branchpoints over fixed points of the z-plane, each branchpoint having the order infinity in every sheet (that is, the simply connected covering surface of the whole plane punctured in n real points of the real axis). This surface is mapped on the upper half plane in such a way that each upper or lower half-plane of the original surface is mapped on a portion of the upper half-plane cut out by n half circles which have their centers on the real axis and touch their two neighbors. *R. Courant* (New York, N. Y.).

Wirtinger, Wilhelm. Translationsmannigfaltigkeiten, welche zu Kurven vom Geschlechte Null oder Eins gehören. Monatsh. Math. Phys. **48**, 30–40 (1939). [MF 623]

The author gives a few examples of manifolds of translation which can be obtained in two different ways, the theory of which he has developed in his previous paper: Lies Translationsmannigfaltigkeiten und Abelsche Integrale [Monatsh. Math. Phys. **46**, 384–431 (1938)]. These examples are interesting due to the fact that the necessary calculations can be accomplished and that, in one case, an algebraic manifold is obtained. *E. Helly.*

Kneser, Hellmuth. Majoranten beim Weierstrassschen Vorbereitungssatz. Monatsh. Math. Phys. **48**, 26–29 (1939). [MF 622]

The author gives an ingenious proof of the Weierstrass preparation theorem: If

$$f(x, y_1, \cdots, y_n)=x^m-\sum a_{\mu;(\nu)}x^{\mu}y_1^{\nu_1}\cdots y_n^{\nu_n}$$

is a convergent power series (in the neighborhood of the origin) with $a_{j;(0)}=0$ for $j=0, 1, \cdots, m$, then there exists a unique convergent power-series

$$g(x, y_1, \cdots, y_n)=1+\sum b_{\mu;(\nu)}x^{\mu}y_1^{\nu_1}\cdots y_n^{\nu_n},$$

with $b_{0;(0)}=0$, such that the product $f\cdot g$ has the form

$$x^m+\sum_{\mu<m} c_{\mu;(\nu)}x^{\mu}y_1^{\nu_1}\cdots y_n^{\nu_n}.$$

The method is the classical method (in differential equations and elsewhere) of undetermined coefficients and majorants of power-series. *W. T. Martin.*

Jung, Heinrich W. E. Zur Theorie der algebraischen Funktionen zweier Veränderlicher. II. J. Reine Angew. Math. **181**, 68–82 (1939). [MF 589]

The object of the paper, as stated in the introduction, is to show how the simple integrals of the second kind attached to an algebraic surface can be obtained by parameter differentiation of simple integrals of the third kind. With this object as guide, the reader is ill prepared to find later an exposition of an algebro-geometric proof of the fundamental theorem of the theory of surfaces concerning the completeness of the characteristic series of a complete continuous system of curves on a surface. This proof, if correct, would have been the first of its kind. However, the proof is false, since it is based upon the following false assertion: if $f_i(\mu_1, \cdots, \mu_\sigma; \lambda_1, \cdots, \lambda_\sigma)$, $i=1, 2, \cdots, \delta$; $\delta\geqq\sigma$, are holomorphic functions at the initial values $\mu=\lambda=0$, and if the equations $f_i=0$ have a unique solution for $\mu_1, \cdots, \mu_\sigma$ when the λ's are sufficiently small, then the Jacobian

$$\left|\frac{\partial f_i}{\partial \mu_j}\right|_{\lambda=\mu=0}, \quad i=1, 2, \cdots, \delta; j=1, 2, \cdots, \sigma,$$

has rank σ [see equations (21)]. *O. Zariski.*

Bergmann, Stefan. Über die Kernfunktion gewisser Reinhardtscher Kreiskörper. Rev. Math. Union Interbalkan. **2**, 41–43 (1939). [MF 595]

Im Raume zweier komplexen Veränderlichen w, z sei ein Rheinhardtscher Körper $z^2<G(|w|^2)$, $0\leqq|w|<1$, gegeben. Die zugehörige Kernfunktion lässt sich dann in der Form $K=\sum_{\rho=0}^{\infty}E_\rho(|w|^2)|z|^{2\rho}$ darstellen. Verfasser beweist nun, dass, falls die Randfunktion $G(R)$, $R=|w|^2$, des gegebenen Körpers ein Polynom $n$ten Grades in R ist, die Koeffizienten $E_\rho(R)$ der Kernfunktion einer gewöhnlichen Differentialgleichung $n(\rho+1)$ten Ordnung genügen, deren Koeffizienten in einfacher Weise von G abhängen. [Über die Begriffe: Rheinhardtscher Körper, Kernfunktion, etc., siehe Behnke-Thullen: Theorie der Funktionen mehrerer komplexen Veränderlichen, Ergebnisse Math. III$_3$, Springer, Berlin, 1934]. *P. Thullen* (Quito).

Fueter, Rud. Über einen Hartogs'schen Satz. Comment. Math. Helv. **12**, 75–80 (1939). [MF 501]

Verfasser beweist aus den Elementen der Theorie der rechtsregulären Funktionen einer Quaternionenvariablen z den folgenden Hauptsatz: Ist $w=f(z)$ eine auf der geschlossenen endlichen, zweiseitigen, sich nirgends durchdringenden Hyperfläche R eindeutige und analytische rechtsreguläre Funktion, so ist $f(z)$ auch im ganzen Innern von R analytisch rechtsregulär. Da insbesondere eine analytische Funktion zweier komplexen Veränderlichen als eine analytische rechtsreguläre Funktion einer Quaternionenvariablen aufgefasst werden kann, folgt sofort als Spezialfall der bekannte Hartogs'sche Satz, dass eine auf einer solchen Hyperfläche analytische Funktion zweier komplexen Veränderlichen auch im ganzen Innern des von der Hyperfläche begrenzten Bereiches analytisch ist. *P. Thullen* (Quito).

Fueter, Rud. Über vierfachperiodische Funktionen. Monatsh. Math. Phys. **48**, 161–169 (1939). [MF 636]

Verf. erweitert den Begriff der vierfachperiodischen Funktionen zweier komplexen Veränderlichen (Abelschen Funktionen) durch die Einführung der vierfachperiodischen (meromorphen) rechtsregulären Funktionen einer Quaternionenvariablen, die er wie folgt definiert: Eine rechtsreguläre Funktion einer Quaternionenvariablen heisse "vierfachperiodisch," wenn sie (1) im ganzen endlichen R_4 nur unwesentliche Singularitäten besitzt und (2) vier Perioden w_l, $l=1, 2, 3, 4$, hat, deren zugehörige Komponentendeterminante nicht verschwindet. Eine Abelsche Funktion ist im besonderen eine rechtsreguläre vierfachperiodische Funktion; aber auch umgekehrt ist ein rechtsreguläre vierfachperiodische Funktion stets eine Abelsche, wenn nur zwei ihrer vom Realteil verschiedenen Komponenten identisch verschwinden. In Analogie zu den Weierstrass'schen $\wp$-Funktionen stellt Verf. gewisse spezielle vierfachperiodische Funktionen $\wp_{n_1n_2n_3}(z)$ auf, mit deren Hilfe sich jede beliebige vierfachperiodische Funktion darstellen lässt; dies unter einer gewissen Voraussetzung, die Verf. in einer nächsten Arbeit beweisen wird. Unter der gleichen Voraussetzung gilt der Satz, dass "eine vierfachperiodische Funktion, die im Periodenparallelotop überall regulär ist, mit Ausnahme eines isolierten Punktes, in dem sie höchstens eine Singularität dritter Ordnung hat, eine Konstante ist."
P. Thullen (Quito).

Popoff, Kyrille. Sur une extension de la notion de dérivée. Fonctions de variables complexes. C. R. Acad. Sci. Paris **209**, 472–474 (1939). [MF 516]

In analogy to his earlier work on real variables [C. R. Acad. Sci. Paris **206**, 1444 (1938), and **207**, 110 (1938)] the author gives a generalization of the derivative of a function $f(z)=P(x, y)+iQ(x, y)$ by use of a certain minimizing process. The generalized derivative exists and is equal to $\partial P/\partial x+i\partial Q/\partial x$ whenever P and Q satisfy the Cauchy-Riemann equations, but it may also exist whenever P and Q have discontinuities on an everywhere dense set of measure zero. *W. T. Martin* (Cambridge, Mass.).

Popoff, Kyrille. Nouvelle extension de la notion de dérivée. Fonctions de variables complexes. C. R. Acad. Sci. Paris **209**, 668–670 (1939). [MF 711]

The author defines the derivative of a function as the limit of the ratio of a certain pair of double integrals. This definition furnishes a generalization of the derivative of an analytic function and is somewhat simpler than his previous generalization [C. R. Acad. Sci. Paris **209**, 472–474 (1939)]. *W. T. Martin* (Cambridge, Mass.).

Poor, Vincent C. On circulation functions. Amer. J. Math. **61**, 833–842 (1939). [MF 281]

The author calls $f(z)$, $z=x+iy$, a circulation function in a region D if its areal derivative $\partial f/\partial\alpha$ exists for every $z_0\varepsilon D$ ($\partial f/\partial\alpha=\lim 1/(2i\sigma)\int_C f(z)dz$, where C is a closed rectifiable Jordan curve containing z_0 in its interior σ, and where C is being deformed continuously into z_0). Pompeiu gave an integral representation of $f(z)$ analogous to the Cauchy integral formula for analytic functions:

$$(*) \qquad f(u)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-u}dz-\frac{1}{\pi}\iint_\sigma\frac{\partial f}{\partial\alpha}\frac{d\sigma}{z-u}$$

($u\varepsilon\sigma$, $C\subset D$, C boundary of σ). In the present paper the inverse question is treated: Given two continuous functions, $F(z)$ defined on C, $\varphi(z)=\partial F/\partial\alpha$ defined on σ, when does the right-hand side of (*), with f replaced by F, represent a circulation function in σ which (i) has the areal derivative $\varphi(z)$ for every $z\varepsilon\sigma$, and (ii) assumes the boundary values $F(z)$ as $u\varepsilon\sigma$ approaches a point $z\varepsilon C$ at which C has a tangent? The main result is: A necessary and sufficient condition that (*) represent such a function $f(u)$, for $u\varepsilon\sigma$, is that

$$\frac{1}{2\pi i}\int_C \frac{F(z)}{z-x}dz-\frac{1}{\pi}\int\int_\sigma \frac{\varphi(z)d\sigma}{z-x}=\tfrac{1}{2}F(x), \qquad x\varepsilon C,$$

for every x where C has a tangent (that is, almost everywhere). The first integral here is meant to be taken as a principal value in a certain sense defined by the author. A similar result for circulation functions in the exterior of C and extensions to "circulation functions of second kind" (defined in the paper) are studied. *S. E. Warschawski.*

Special Functions

Pasternack, Simon. A generalization of the polynomial $F_n(x)$. Philos. Mag. **28**, 209–226 (1939). [MF 872]

This generalization of $F_n(x)$ is defined by the operational equation

$$F_n^m(D)\operatorname{sech}^{m+1}x=\operatorname{sech}^{m+1}xP_n(\tanh x),$$

where $n+1$ is a positive integer and m can have any value except -1. Two physical applications of the polynomial are indicated and then its properties are developed with the aid of a generating function expressible by means of the hypergeometric series. The value of $F_n^m(0)$ is found in terms of Gamma functions with the aid of Whipple's formula in the theory of generalized hypergeometric functions. Another form of operational equation is derived and various series are obtained. Recurrence formulae are obtained and definite integrals are found for the representation of $F_n^m(z)$. Two more generating functions are found and the polynomial is found to occur also in an expansion of $(u+1)^k(u-1)^{-k-1}$ in a series of associated Legendre functions $Q_n^m(u)$. A generalization is given to non-integral values of n and the cardinal formula of interpolation is found to hold for $|\Re(z+1)|<\Re(m)$. The paper closes with a discussion of the modes of vibration of a rotating string of weights.

H. Bateman (Pasadena, Calif.).

Toscano, Letterio. Formule limiti sui polinomi di Laguerre. Boll. Un. Mat. Ital. **1**, 337–339 (1939). [MF 576]

The nth Hermite polynomial $H_n(x)$ is expressed as a limit of Laguerre polynomials $L_n^{(\alpha)}(x)$

$$H_n(x)=2^n n!\lim_{\omega\to\infty}\omega^{-n}L_n^{(\alpha)}(\omega x+\tfrac{1}{2}\omega^2),$$

where $\alpha=\frac{1}{2}\omega^2+k$, k arbitrary. Comparison with classical formulas giving Bessel functions as limits of Laguerre polynomials. *E. Hille* (New Haven, Conn.).

Meijer, C. S. Ueber Produkte von Legendreschen Funktionen. Nederl. Akad. Wetensch., Proc. **42**, 930–937 (1939). [MF 978]

This is, in a way, the continuation of a former paper of the same author [Nieuw Arch. Wiskde (2) **19**, 207–234 (1938)] and, like the former paper, it contains a great number of integral representations for products of Legendre functions. Functions of both first and second kind are taken into consideration. A typical example is the integral representation

$$\{P_n^m(1+z^2)^{\frac{1}{2}}\}^2=\frac{2z^{-2m}\Gamma(\frac{1}{2}-m)}{\pi^{\frac{1}{2}}\Gamma(-m-n)\Gamma(1-m+n)}\times\int_0^\infty(z^2+\cosh^2 t)^{m-\frac{1}{2}}P_n^m(\cosh 2t)\sinh^{1-m}t\cosh^m t\,dt,$$

valid subject to the conditions

$$z\neq 0,\quad |\arg z|<\tfrac{1}{2}\pi,\quad \Re(m+n)<0,\quad \Re(m-n)<1.$$

A. Erdélyi (Edinburgh).

Howell, W. T. A note on Laguerre polynomials. Philos. Mag. **28**, 287–288 (1939). [MF 1064]

Author proves

$$\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}e^{-(x+y)}L_m^{(\alpha)}(x)L_n^{(\alpha)}(y)=\frac{2^{2\alpha+1}}{\pi}\int_0^{\pi/2}\int_{-\pi/2}^{\pi/2}\sin^{2n+2\alpha+1}\theta\cos^{2m+2\alpha+1}\theta\times\exp(-F)L_{m+n+1}^{(2\alpha)}(F)\cos^{2\alpha}\phi\,d\phi\,d\theta,$$

where $F=2\cos\phi(x\cos^2\theta e^{i\phi}+y\sin^2\theta e^{-i\phi})$, and also a second similar integral representation of a product of Laguerre polynomials. *A. Erdélyi* (Edinburgh).

Howell, W. T. Some formulae for the product of two Whittaker functions with different arguments. Philos. Mag. **28**, 493–495 (1939). [MF 1066]

Proof of

$$e^{\frac{1}{2}(x-y)}W_{k,m}(x)W_{k,m}(y)=\frac{2(xy)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)}\times\int_0^\infty\exp\left\{-x\cosh^2\frac{u}{2}-y\sinh^2\frac{u}{2}\right\}\times K_{2m}((xy)^{\frac{1}{2}}\sinh u)\coth^{2k}\frac{u}{2}du,$$

$$\Re(\tfrac{1}{2}\pm m-k)>0;\quad |\arg x|<\tfrac{1}{2}\pi;\quad |\arg y|<\tfrac{1}{2}\pi,\quad |\arg(x+y)|<\tfrac{1}{2}\pi,$$

and of the corresponding integral representations for the products $W_{k,m}(x)M_{-k,m}(y)$ and $M_{k,m}(x)M_{-k,m}(y)$.

A. Erdélyi (Edinburgh).

Erdélyi, Artur. Integraldarstellungen für Produkte Whittakerscher Funktionen. Nieuw Arch. Wiskde **20**, 1–38 (1939). [MF 564]

Let $M_{k,m}(z)$ and $W_{k,m}(z)$ denote Whittaker's functions [see Modern Analysis, 1927, chap. 16]. The author obtains integral representations for the products $\Pi_{\nu=1}^n M_{k_\nu,m_\nu}(z_\nu)$ and $\Pi_{\nu=1}^n W_{k_\nu,m_\nu}(z_\nu)$ involving certain generalized hypergeometric series of several variables. Products of mixed character are also considered. The reproduction of the complicated formulas seems difficult in a short review. The paper contains a good survey of the known special cases of these formulas, involving Laguerre, Hermite and Charlier polynomials, Bessel and spherical functions, elliptic integrals, and so on.

G. Szegö (Stanford University, Calif.).

Erdélyi, A. Transformation of a certain series of products of confluent hypergeometric functions. Applications to Laguerre and Charlier polynomials. Compositio Math. 7, 340–352 (1939). [MF 800]

The following identities are proved:

$$\sum_{r=0}^{\infty} \alpha_r {}_1F_1(a; c+r; x)\, {}_1F_1(b; d+r; y) z^r$$
$$= e^z \sum_{r=0}^{\infty} \beta_r {}_1F_1(a+r; c+r; x-z)\, {}_1F_1(b+r; d+r; y-z) z^r$$
$$= \exp\left(x+y-\frac{xy}{z}\right) \sum_{r=0}^{\infty} \gamma_r \left(\frac{(x-z)(y-z)}{z}\right)^r$$
$$\times {}_2F_1\left(-r, a; c; \frac{z}{z-y}\right) {}_2F_1\left(-r, b; d; \frac{z}{z-x}\right),$$

where the coefficients α_r, β_r, γ_r depend on a, b, c, d and r. These identities are applied to the Laguerre and Charlier polynomials. Regarding the relationship between these two classes of polynomials, see the reviewer's book: Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., vol. 23, p. 34 (2.81.6). *G. Szegö* (Stanford University, Calif.).

Erdélyi, A. Transformation of hypergeometric integrals by means of fractional integration by parts. Quart. J. Math., Oxford Ser. **10**, 176–189 (1939). [MF 598]

By use of fractional integration by parts, the functional equation

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)} \int_0^1 x^{\lambda-1}(1-x)^{\gamma-\lambda-1} F(\alpha, \beta; \lambda; xz)\, dx,$$
$$\Re\gamma > \Re\lambda > 0;\ z \neq 1;\ |\arg(1-z)| < \pi,$$

is derived directly from the hypergeometric series. The same method is applied for the derivation of a more complicated functional equation in which a product of two hypergeometric functions appears under the integral sign. Several consequences of this formula are considered, also analogous relations between confluent hypergeometric functions. Finally an integral representation of F is obtained involving a generalized hypergeometric function of the type ${}_3F_2$. *G. Szegö* (Stanford University, Calif.).

Belardinelli, G. Un esempio di serie ipergeometrica di ordine infinito. Monatsh. Math. Phys. **48**, 381–388 (1939). [MF 654]

One root of the transcendental equation

$$\theta(y) + xy^p = 0, \qquad p = 0, 1, 2, \cdots,$$

in which $\theta(y)$ is assumed to be an analytic function of y, regular in the neighborhood of $y=0$, and $y=\omega \neq 0$ a simple zero of $\theta(y)$ in the circle of convergence of the expansion

$$\theta(y) = \sum_{n=0}^{\infty} a_n y^n,$$

is represented by the infinite series

$$y = \omega + \sum_{n=1}^{\infty} \frac{(-x)^n}{2\pi i n} \int_\rho \{y^p/\theta(y)\}^n dy,$$

ρ being a suitable contour encircling $y=\omega$ and no other zero of $\theta(y)$. This series represents an analytic function of x, regular in the vicinity of $x=0$. This function is shown to be what author [Boll. Un. Mat. Ital. A. **13**, no. 3 (1934)] calls a hypergeometric function of infinite order.

A. Erdélyi (Edinburgh).

MacRobert, T. M. Formulae for generalized hypergeometric functions as particular cases of more general formulae. Philos. Mag. **28**, 488–492 (1939). [MF 1063]

It is shown how certain known formulae for generalized hypergeometric functions can be derived as particular cases of formulae of more general type involving multiple series. For instance, author proves that when z lies within the inner loop of the curve $16(x^2+y^2) = \{(x-1)^2+y^2\}^2$, then

$${}_4F_3\left(\begin{matrix}\alpha, \beta, \gamma, \delta; z \\ \alpha-\beta+1, \alpha-\gamma+1, \omega\end{matrix}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}\alpha)_n (\frac{1}{2}+\frac{1}{2}\alpha)_n (1+\alpha-\beta-\gamma)_n (\delta)_n}{n!(\alpha-\beta+1)_n(\alpha-\gamma+1)_n(\omega)_n} (-4z)^n$$
$$\times F\left(\begin{matrix}\delta+n, \alpha+2n; z \\ \omega+n\end{matrix}\right),$$

where, for example, $(\delta)_n = \delta(\delta+1)\cdots(\delta+n-1)$, and shows that from this result with $\alpha=-m$, $z=1$ and $\delta=-m$, $z=1$, respectively, $m=0, 1, 2, \cdots$ two formulae of Whipple [Proc. London Math. Soc. (2) **26**, 266, 265 (1926)] transforming nearly poised ${}_4F_3$ into Saalschützian ${}_5F_4$ follow. Similarly Saalschütz's theorem, certain transformations of Saalschützian series and two formulae for terminating well-poised series are dealt with. *A. Erdélyi* (Edinburgh).

Birindelli, Carlo. Sul comportamento asintotico degli integrali generali di due classiche equazioni differenziali. Ist. Lombardo, Rend. **72**, 455–469 (1939). [MF 938]

The author gives a discussion of the asymptotic behavior for large values of the variable of various solutions of the differential equations of Hermite-Weber and Laguerre, the discussion of the latter case being reduced to that of the former. Confluent hypergeometric integrals are discussed by the methods of Whittaker & Watson's Modern Analysis, but no reference is made to this treatise, nor does the author refer to the other literature on this question.

E. Hille (New Haven, Conn.).

Varma, R. S. Some infinite integrals involving parabolic cylinder functions. J. Math. Pures Appl. **18**, 157–166 (1939). [MF 381]

The author starts from a known expression for the integral

$$\int_0^\infty x^m \exp(-x^2/4) D_n(x)\, dx,$$

where D_n is a parabolic cylinder function of order n. By using the power series expansion for the Bessel function J_n of the first kind of order n and by inverting integration and summation, he obtains an expression for the integral

$$\int_0^\infty x^l \exp(-x^2/4) D_m(x) J_n(ax)\, dx.$$

By using the power series for the Bessel function product $J_n J_m$ an expression is obtained in the same way for the integral

$$\int_0^\infty x^l \exp(-x^2/4) D_p(x) J_m(x) J_n(x)\, dx.$$

Starting from the transformation

$$\int_0^\infty (xt)^{\frac{1}{2}p-\frac{1}{2}n+1} J_{\frac{1}{2}n+\frac{1}{2}p}(xt) R_p\, dt = R_n,$$

he calculates the integral

$$\int_0^\infty t^{\lambda+\nu} \exp(t^2/4) D_{-2\mu}(t) J_{\lambda+\mu}(xt)\,dt$$

and discusses some particular cases. Taking R equal to $x^{\rho+\lambda}\exp(-x^2/2)T_n^{\rho}(x^2)$, where T_n is Sonine's polynomial of order n, he obtains an expression for the integral

$$\int_0^\infty t^{\lambda(\rho+\mu+2)} \exp(-t^2/2) J_{\lambda\mu+\lambda\nu}(xt) T_\mu^{\rho}(t^2)\,dt$$

in terms of Kummer's function. Finally he calculates the integral

$$\int_0^\infty \frac{t^{2\mu+1}\exp(-x^2t^2/2)T_\mu^{\rho}(x^2t^2)}{(1+t^2)^{\lambda\mu+\lambda\nu+1}}\,dt.$$

M. J. O. Strutt (Eindhoven).

Meijer, C. S. Ueber Besselsche, Lommelsche und Whittakersche Funktionen. (Erste Mitteilung.) Nederl. Akad. Wetensch., Proc. **42**, 872–879 (1939). [MF 704]

Proof of the integral formula

$$\pi i\, {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;\zeta) = \Gamma(\alpha)\Gamma(\beta)\int_{-i\infty}^{+i\infty} I_{\alpha-\beta}(2t)$$

$$\times {}_{p+2}F_q(\alpha,\beta,a_1,\cdots,a_p;b_1,\cdots,b_q;\zeta/t^2)t^{-\alpha-\beta+1}\,dt,$$

valid for $0\leqq p\leqq q-1$; α and β are complex quantities satisfying $\Re(\alpha+\beta)>\frac{1}{2}$, $\alpha\neq 0,-1,-2,\cdots$; $\beta\neq 0,-1,-2,\cdots$, but otherwise arbitrary. The path of integration is curved so that $t=0$ lies on the left of the path. If $p=q-1$ then the path of integration is so curved that in every point $|t|>|\xi^{\frac{1}{2}}|$. Special values of the parameters occurring in the quoted formula yield similar relations with Bessel functions and products of Bessel functions. *A. Erdélyi.*

Meijer, C. S. Ueber Besselsche, Lommelsche und Whittakersche Funktionen. (Zweite Mitteilung.) Nederl. Akad. Wetensch., Proc. **42**, 938–947 (1939). [MF 979]

This paper contains two integral representations for $H_\mu^{(1)}(z)H_\nu^{(2)}(z)-H_\nu^{(1)}(z)H_\mu^{(2)}(z)$, an integral representation for $I_{-\mu}(z)I_{-\nu}(z)-I_\mu(z)I_\nu(z)$, also for Lommel's function $r_{\mu,\nu}(z)$ and integral representations for products of confluent hypergeometric functions $W_{k,m}(z)W_{k,m}(-z)$, $W_{k,m}(z)W_{-k,m}(z)$, $M_{k,m}(z)M_{k,m}(-z)$, $W_{k,m}(z)\{W_{k,m}(ze^{\pi i})-W_{k,m}(ze^{-\pi i})\}$. A typical example is

$$W_{k,m}(z)W_{-k,m}(z) = 2^{m+\frac{1}{2}}\pi^{-\frac{1}{2}}z^{-m+\frac{1}{2}}\int_0^{\frac{1}{2}\pi} K_{m+\frac{1}{2}}(z\sec\varphi)$$

$$\times P_{k-\frac{1}{2}}^{m}(\cos 2\varphi)\sin^{1-m}\varphi\cos^{2m-1}\varphi\,d\varphi,$$

with conditions of validity $\Re(m)<1$, $|\arg z|<\frac{1}{2}\pi$.

A. Erdélyi (Edinburgh).

Jones, R. Clark. On the theory of fluctuations in the decay of sound. J. Acoust. Soc. Amer. **11**, 324–332 (1940). [MF 974]

In analytical terms, the author is concerned with a random walk in two dimensions, where all directions are equally probable and the length of the nth step is p_n. K. Pearson and Watson have shown that the probability $P_n(r)$ that the moving point is, after n steps, at a distance less than r from the origin is

$$P_n(r) = r\int_0^\infty J_1(rx)J_0(p_1x)\cdots J_0(p_nx)\,dx$$

[cf. G. N. Watson, Theory of Bessel Functions, Cambridge, England, 1922, p. 420]. The main object of the present paper is to find a usable approximation to dP_n/dr; the method is a generalization of the procedure employed by Pearson in the case of equal p_k. The results are applied to a statistical investigation of the fluctuations in the decay of sound in a room which are caused by interference of the various natural modes of vibration. *W. Feller.*

Giulotto, Virgilio. Polinomi di Hermite e di Didon nel dominio iperarmonico. Ist. Lombardo, Rend. **72**, 37–57 (1939). [MF 941]

If Δ denotes the Laplacian operator $\sum\partial^2/\partial z_k^2$ in an arbitrary number of variables, and if $\Delta^1F=\Delta F$, $\Delta^qF=\Delta(\Delta^{q-1}F)$, a function is said to be hyperharmonic, or more specifically q-harmonic if it satisfies the differential equation $\Delta^qF=0$. Certain homogeneous q-harmonic polynomials in $n+2$ variables $z_1,\cdots,z_{n+2}$ are constructed which, when expressed in terms of "zonal" coordinates $r=(z_1^2+\cdots+z_{n+2}^2)^{\frac{1}{2}}$, $x_1,\cdots,x_n$, ω, are independent of ω and reduce on the hypersphere $r=1$ to polynomials in $x_1,\cdots,x_n$. These generalize the polynomials corresponding to the case $q=1$ introduced from a different point of view by Hermite for $n=2$ and by Didon for arbitrary n. A relation of recurrence is derived. A second set of polynomials is defined corresponding to those which for $q=1$ are constructed to form a biorthogonal system with the first set; but the property of biorthogonality is not discussed for the new polynomials. For comparison with the previously existing literature reference is made particularly to P. Appell and J. Kampé de Fériet, Fonctions hypergéométriques et hypersphériques—Polynomes d'Hermite, 1926. *D. Jackson* (Minneapolis, Minn.).

Differential Equations

Hebroni, P. Über lineare Differentialgleichungen in Ringen und ihre Anwendungen auf lineare Integrodifferentialgleichungen. III. Compositio Math. **7**, 229–252 (1939). [MF 797]

The author's abstract theory of linear differential equations of the first order [Compositio Math. **5**, 403–429; **6**, 258–284] is extended to systems, homogeneous or nonhomogeneous, of r linear differential equations of the first order in r unknowns. Applications are made to a class of linear integrodifferential equations. *J. F. Ritt.*

Hebroni, P. Sur les équations différentielles linéaires dans un anneau de certaines matrices continuisées (matrices doubles D_1^2) et leurs applications à certaines équations intégrodifférentielles. C. R. Acad. Sci. Paris **209**, 712–714 (1939). [MF 801]

Attention is called, briefly, to a new concrete specialization of the author's abstract theory of linear differential equations. *J. F. Ritt* (New York, N. Y.).

Kolchin, E. R. On the basis theorem for infinite systems of differential polynomials. Bull. Amer. Math. Soc. **45**, 923–926 (1939). [MF 784]

Given an infinite system Σ of differential polynomials and a basis Φ of Σ, the author describes Φ as strong if there exists a positive integer p such that, for every form G in Σ, the form G^p is in the differential ideal generated by Φ. What is noteworthy in the definition is that a single integer p takes care of every G in Σ. Raudenbush has shown [Bull.

Amer. Math. Soc. **42**, 371–373 (1936)] that a basis may fail to be strong. Deepening the work of Raudenbush, the author shows that the perfect ideal generated by the form uv in the unknowns u and v does not have even a single strong basis. *J. F. Ritt* (New York, N. Y.).

de Losada y Puga, Cristóbal. On the theory of singular solutions of differential equations of the first order. (Methodological note). Revista Univ. Católica Peru **8**, 521–525 (1939). (Spanish). [MF 981]

Vietoris, L. Über die Integration gewöhnlicher Differentialgleichungen durch Iteration. III. Monatsh. Math. Phys. **48**, 19–25 (1939). [MF 621]

Se $y(x)$ é la soluzione dell'equazione $y'=f(x, y)$ verificante la condizione $y(a)=b$ e $y_0(x), y_1(x), \cdots, g_0(x, y), g_1(x, y), \cdots$ sono due successioni di funzioni soddisfacenti le equazioni

$$f(x, y_m(x))=g_m(x, y_m(x)), \quad y'_{m+1}=g_m(x, y_{m+1}), \quad y_m(a)=b,$$

si ha, uniformemente in un conveniente intorno di a, $\lim_{m\to\infty} y_m=y$, le funzioni $f(x, y)$ e $g_m(x, y)$ essendo supposte lipschitziane con coefficienti di Lipschitz equilimitati. In tale teorema é contenuta la dimostrazione della convergenza di vari procedimenti di iterazione per il calcolo di $y(x)$, considerati dall'autore in due precedenti note [Monatsh. Math. Phys. **39**, 15–20 (1932); **41**, 384–391 (1934)]. Un teorema analogo sussiste per i sistemi di equazioni differenziali del primo ordine. *C. Miranda* (Torino).

Kun Kuti, Márton. Über die homogenen linearen Differentialgleichungen, deren sämtliche Lösungen ganze Funktionen sind. Mat. Fiz. Lapok **46**, 152–169 (1939). (Hungarian) [MF 987]

One of the results of this paper is: Suppose that in the differential equation $\sum_{\nu=0}^{n} P_\nu(z)y^{(n-\nu)}=0$, the P_ν are analytic functions, $P_0\equiv 1$, P_1 an entire function. Then all solutions are entire functions if and only if all P_ν are entire functions. *O. Szász* (Cincinnati, Ohio).

Germay, R. H. J. Sur une propriété résultant de l'interversion des valeurs initiales et des variables indépendantes dans certaines fonctions associées aux solutions des systèmes complètement intégrables d'équations aux différentielles totales à coefficients linéaires. Bull. Sci. Math. **63**, 272–278 (1939). [MF 972]

The author considers the system

$$dz_j=\sum_{k=1}^{n} a_{jk}(x_1, x_2, \cdots, x_n; z_1, z_2, \cdots, z_m)dx_k, \quad j=1, 2, \cdots, m,$$

with the initial conditions $z_j(x_1^0, x_2^0, \cdots, x_n^0)=z_0^j$. He derives for the solution the formula

$$z_j=z_j^0+H_{j0}+\sum_{s=1}^{m} z_s^0 H_{js},$$

in which the H_{js} are functions of $x_1, x_2, \cdots, x_n$ and $x_1^0, x_2^0, \cdots, x_n^0$, the definition of which would be too long for this review. The main object of the paper is the proof of certain formulas which express the effect of the interchange of the independent variables x_i with the initial values x_i^0 in the functions H_{js}. *E. Rothe* (Oskaloosa, Iowa).

de Rham, Georges. Sur un procède de formation d'invariants intégraux. Jber. Deutsch. Math. Verein. **49**, 156–161 (1939). [MF 685]

The author considers an analytic manifold E of n dimensions, and an analytic, transitive Lie group G of transformations of E. An integral

$$\int_{C^p}\omega=\int_{C^p}\sum A_{i_1\cdots i_p}dx_{i_1}\cdots dx_{i_p} \tag{1}$$

is called an integral invariant of order p if it is unaltered in value when its p-dimensional domain of integration C^p is transformed by G. If 0 is a point of E, it is supposed that the subgroup g of G which leaves 0 fixed is compact ("clos"). The main result of the paper is the following: If C^p, C^{n-p} are domains of integration, of dimension as indicated, the Kronecker index $I(C^p, xC^{n-p})$ of C^p and the transform of C^{n-p} by an element x of G, is integrable over G, and the function

$$F(C^p, C^{n-p})=\int_G I(C^p, xC^{n-p})dx$$

is an integral invariant of order p, depending only on C^{n-p}:

$$F(C^p, C^{n-p})=\int_{C^p}\omega(C^{n-p}).$$

Moreover, all integral invariants of type (1) can be generated in this manner, which is distinct from the infinitesimal method of E. Cartan [Ann. Soc. Polon. Math. **8**, 181–225 (1929)]. The method yields a new proof of a known theorem of Cartan. *A. E. Taylor* (Los Angeles, Calif.).

Piaggio, H. T. H. Geometry of special integrals of Lagrange's equation. J. London Math. Soc. **14**, 228–239 (1939). [MF 247]

This paper gives only a geometrical interpretation of special integrals of Lagrange's partial differential equation

$$P(x, y, z)p+Q(x, y, z)q=R(x, y, z),$$

whose analytical aspect has been dealt with in a previous paper by the author [J. London Math. Soc. **13**, 264–272 (1938)]. In the last paragraph the author deals with the identification of the special integrals of a linear partial differential equation that can be rationalized with the singular integral of the rationalized form, or with the focal surface of the congruence formed by the characteristics.
A. Kawaguchi (Sapporo).

Chaundy, T. W. Hypergeometric partial differential equations (III). Quart. J. Math., Oxford Ser. **10**, 219–240 (1939). [MF 601]

The differential equation

$$f(\delta)V=x^a y^b g(\delta')V, \qquad \delta, \delta'\equiv x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y},$$

can be reduced, by a suitable transformation, to the simpler form $y^m f(\delta)V=x^m g(\delta')V$, where f and g are supposed to be polynomials of equal degree m, the leading coefficients in both being unity. For this differential equation the solution of Cauchy's problem in terms of suitable defined "fundamental solutions" or "Green's functions" is found. The fundamental solutions are explicitly obtained and fully discussed in the case of the equation of third order which can be reduced to the form

$$y^3\delta(\delta+3p_1+1)(\delta+3p_2+2)V=x^3\delta'(\delta'+3q_1+1)(\delta'+3q_2+2)V,$$

in which $p_1, \cdots, q_2$ first are supposed to be positive integers. The fundamental solutions obtained thus are transformed into contour integrals which are valid for any values of $p_1, \cdots, q_2$. The analogous second-order equation is also

dealt with. The technique used is similar to that of an earlier paper of the same author [Quart. J. Math., Oxford Ser. 8, 280–302 (1937)]. *A. Erdélyi* (Edinburgh).

Barrer, R. M. Gas flow in solids. Philos. Mag. **28**, 148–162 (1939). [MF 1122]

The determination of different physical constants is mathematically reduced to a study of the solution $w(t, x)$ of $\partial w/\partial t = D\partial^2 w/\partial x^2$ in $0<x<1$ with $w(0, x)=f(x)$ and the boundary conditions $\pm\partial w/\partial x = kw$ at $x=0$ and $x=l$. (These conditions are misprinted in the original.) *W. Feller.*

Awbery, J. H. The periodic flow of heat in a hollow cylinder. Philos. Mag. **28**, 447–451 (1939). [MF 1067]

Starting with the differential equation of heat conduction transformed to cylindrical-polar coordinates, the problem of the flow of heat in a hollow cylinder of infinite length is investigated supposing that the temperature of the inner wall is constrained to vary periodically and that the heat loss of the outer surface is proportional to its temperature excess over the surroundings. The various Bessel functions occurring in the solution are complex valued. The formulae for amplitude and phase of the waves are expressed in terms of tabulated, real-valued functions only.

A. Erdélyi (Edinburgh).

Howell, W. T. A note on the solution of some partial differential equations in the finite domain. Philos. Mag. **28**, 396–402 (1939). [MF 1065]

Discussion of the solutions of following equations:

$$\frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial^2 w}{\partial z^2} + a^2c^2w = 0,$$

$$\frac{\partial V}{\partial t} - h^2\frac{\partial^2 V}{\partial x^2} = 0,$$

$$\frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial^4 w}{\partial x^4} = 0,$$

$$\frac{\partial^n}{\partial x^n}\frac{\partial^2 w}{\partial t^2} - \sigma^2\frac{\partial^n w}{\partial x^n} - \sigma^2\frac{\partial^{n-2} w}{\partial x^{n-2}} = 0,$$

$$\frac{\partial^n}{\partial x^n}\frac{\partial^2 w}{\partial t^2} - \sigma^2\frac{\partial^{n-1} w}{\partial x^{n-1}} = 0,$$

the initial conditions being given. Laplace transform, applied with respect to t, transforms these equations into ordinary linear differential equations which in this paper are solved by Boole's operational method. Thus for the solutions of the partial differential equations formulae are obtained in which the boundary values do not appear explicitly. *A. Erdélyi* (Edinburgh).

Cerf, G. Les équations linéaires aux dérivées partielles et la méthode de la variation des constantes. J. Math. Pures Appl. **18**, 291–302 (1939). [MF 808]

This paper presents a treatment of Laplace's equation

$$(1) \qquad z_{xy} + az_x + bz_y + cz = 0,$$

related to the treatment of Darboux, Théorie des surfaces, Livre 4, Chap. 8. Consider the expression

$$(2) \qquad z = \sum_{i=1}^{5} \lambda_i z_i,$$

where the z_i are particular solutions of (1). We propose to determine the λ_i as functions of x and y so that (2) will be a solution of (1). Arguments involving the counting of arbitrary constants and conditions of integrability show that each of the λ_i must satisfy an equation of the form (1). The problem of determining the λ_i is reduced to the solution of one equation of the form (1) and quadratures. The cases of more or less than five particular solutions z_i are considered.

E. W. Titt (Hyattsville, Md.).

Mathisson, Myron. Le problème de M. Hadamard relatif à la diffusion des ondes. Acta Math. **71**, 249–282 (1939). [MF 766]

The author investigates the solutions $u(x_0; x_1, x_2, x_3)$ of the equation

$$(*) \qquad \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_0^2}\right)u + \sum_0^3 A_i\frac{\partial u}{\partial x_i} + Bu = 0$$

with a view to determining all sets of coefficients A_i, B such that the solution satisfies what Hadamard calls Huyghens' minor premise, that is, such that $u(x_0; x_1, x_2, x_3)$ depends only on the values of $u(0; \xi_1, \xi_2, \xi_3)$ in the neighborhood of the intersection of $\xi_0=0$ and the characteristic cone $x_0^2-(\xi_1-x_1)^2-(\xi_2-x_2)^2-(\xi_3-x_3)^2=0$. Since Huyghens' minor applies to the special case (E) of $(*)$ where $A_i=B=0$, it also applies to those equations $(*)$ in u which can be obtained from (E) by replacing u by λu with an arbitrary function λ. Now the purpose of the paper is to show that all equations $(*)$ satisfying Huyghens' minor can thus be obtained. The method consists in (1) deriving an integral equation for u that reduces directly to the correct resolution of $(*)$ if and only if the minor holds; (2) indicating differential conditions for the A_i and B, necessary for this reduction. These conditions appear as the requirement that a certain function $v(x;\xi)$, determined explicitly from the A_i and B, satisfy the adjoint of $(*)$. To verify, however, that this requirement implies the main theorem of the paper is a lengthy matter necessitating the absolute calculus technique for the discussion of the development of v up to the fourth order. *H. Lewy* (Berkeley, Calif.).

Bourgin, D. G. and Duffin, R. The Dirichlet problem for the vibrating string equation. Bull. Amer. Math. Soc. **45**, 851–858 (1939). [MF 770]

The authors demonstrate that, for special partial differential equations of hyperbolic type, boundary value problems analogous to those of Dirichlet and Neumann, customarily reserved for elliptic equations, can have meaning. In particular, they treat the vibrating string equation (1) $y_{xx}-y_{tt}=0$ in the rectangle $0\leqq x\leqq S$, $0\leqq t\leqq T$. By considerations of the Fourier expansions of the solutions of (1), it is shown that a necessary and sufficient condition for the validity of the customary uniqueness theorems for boundary value problems of this type is that S/T be an irrational number. An existence theorem is given when S/T is an irrational number of a prescribed kind, and boundary values are given satisfying associated prescribed conditions.

J. W. Green (Rochester, N. Y.).

Thomas, T. Y. and Titt, E. W. On the elementary solution of the general linear differential equation of the second order with analytic coefficients. J. Math. Pures Appl. **18**, 217–248 (1939). [MF 805]

Using the methods of the modern tensor analysis, the authors give an elegant exposition of Hadamard's construction of the elementary solution for the equation mentioned

in the title (and supposed to be of elliptic or hyperbolic type). The authors "have made direct use of the invariant form of the equation, the coefficients of which are the components of tensors, the well-known properties of normal coordinates, and have so treated the case where n is even that the convergence proof can be given without resort to Cauchy's existence theorem." In this part the paper is, as the authors state, largely expository. The following interesting theorem is, however, new: A Riemann space R is flat if, and only if, the determinant of the components of the fundamental metric tensor is constant in every system of normal coordinates. (For this, it is sufficient to suppose that the components of the metric tensor have continuous derivatives of order five.) The authors deduce that the Laplace-Beltrami equation defined in R admits the Euclidean solution s^{n-2} or $\log s$ (with s the geodesic distance) if, and only if, R be flat. *W. Feller* (Providence, R. I.).

Pleijel, Aake. Sur les propriétés asymptotiques des fonctions et valeurs propres des plaques vibrantes. C. R. Acad. Sci. Paris **209**, 717–718 (1939). [MF 803]

Recently [Ark. Mat., Astr. Fys. **26**, no. 19; these Rev. **1**, 56 (1940)] the asymptotic distribution of proper values of differential equations, as previously analyzed by H. Weyl and by the reviewer, has been treated on the basis of the new Tauberian theorems. The present note contains the application of this method to the eigenvalue problem of the clamped plate. The result is a new proof for the asymptotic expression of the eigenvalues. *R. Courant.*

Despujols, Pierre. Sur les forces élastiques autour d'une galerie horizontale de section circulaire. C. R. Acad. Sci. Paris **209**, 549–551 (1939). [MF 521]

In connection with the study of the elastic forces set up in the earth due to the presence of a tunnel, this paper considers the system of second order partial differential equations $P_x+Q_y=0$, $P_y-Q_z=0$, where $P=(\lambda+2\mu)(u_x+v_y)$, $Q=\mu(u_y-v_x)$, and λ and μ are constants. The author announces that he is able to satisfy the appropriate boundary conditions with a solution of the form

$$u=c_1\frac{\lambda+\mu}{\mu}\frac{x^3}{r^4}+c_2x+c_4\frac{x}{r^2}+c_5\left(\frac{3x}{r^4}-\frac{4x^3}{r^6}\right),$$

$$v=c_1\left(\frac{\lambda+\mu}{\mu}\frac{x^2y}{r^4}-\frac{y}{r^2}\right)+c_3y+c_4\frac{y}{r^2}+c_5\left(\frac{y}{r^4}-\frac{4x^2y}{r^6}\right).$$

E. W. Titt (Hyattsville, Md.).

Brelot, Marcel. Sur la théorie moderne du potentiel. C. R. Acad. Sci. Paris **209**, 828–830 (1939). [MF 861]

A set E is said to be "thin" [effilé] at a point O, belonging or not to E, if the set $E\cdot CO$ is empty or at a positive distance from O, or is such that there is a neighborhood of O in which there exists a subharmonic function u whose value at O is greater than the superior limit at O of u in $E\cdot CO$. Thus if O is a point of the boundary of a domain Ω the condition that $C\Omega$ be thin at O is that O be an irregular boundary point. So much is evident by the corresponding property of the conductor potential. The author states without proof theorems and criteria which generalize this property and related ones to more general sets. He interprets them with respect to continuous transformations T which increase or decrease all distances, the latter case being instanced in the plane by circular projection onto an axis Ox along circles with center O. If T decreases distances, a set E which is thin at O becomes a set E' which is thin "interiorally" at O', that is, such that every closed subset of $E'+O'$ is thin at O'. *G. C. Evans* (Berkeley, Calif.).

Brelot, M. Familles de Perron et problème de Dirichlet. Acta Litt. Sci. Szeged **9**, 133–153 (1939). [MF 606]

Let Ω be a bounded domain with a function f assigned arbitrarily on its frontier F, not excluding infinite values. Let G_f be the class of functions which are each bounded above and subharmonic in Ω, and have for any point M of F superior limits, on approach to M, not greater than $f(M)$. If the class G_f is not empty, its upper envelope $\underline{H}_f^{\Omega}(P)$ is identically $+\infty$ or harmonic in Ω, and the family G_f contains continuous functions which have the same superior envelope. In fact, the class G_f contains harmonic functions with this same envelope (although on approach to M, if for example M is an irregular boundary point, the harmonic functions may become negatively infinite, even if $f(M)$ is finite). This function is called a hypofunction. Similarly, there exists a hyperfunction, lower envelope of superharmonic functions dominating f. If G_f is empty, the hypofunction is defined to be identically $-\infty$.

If $\underline{H}\equiv\bar{H}$, the resulting function is named "Wiener function" and it is shown that a necessary and sufficient condition that a Wiener function be defined for f is that f be summable on F with respect to the mass distribution $\mu^{\Omega}(e\cdot F)$ resulting from the sweeping out of unit mass from a particular point Q of Ω. In this case

$$\underline{H}_f^{\Omega}(P)=\bar{H}_f^{\Omega}(P)=\int_F f d\mu^P.$$

In the first part of the paper a simple treatment is given of the case where f is bounded. Wiener's theorem [J. Math. Phys. Mass. Inst. Tech. **4**, 21–32 (1925)] is proved, that if f is continuous on F the Wiener function exists and is identical with the sequence solution. Extension is made to general bounded open sets. The paper provides a simple and rather complete treatment of the problems arising from Perron's method of application of subharmonic functions to the Dirichlet problem [Math. Z. **18**, 42–54 (1923)].

G. C. Evans (Berkeley, Calif.).

Nevanlinna, Rolf. Bemerkungen zum alternierenden Verfahren. Monatsh. Math. Phys. **48**, 500–508 (1939). [MF 664]

Let D_1 and D_2 be regions for which the Dirichlet problem may be solved, and let $D=D_1+D_2$; $\bar{\alpha}$ is the portion of the boundary of D_1 within D_2 and α is the remainder of that boundary; $\bar{\beta}$ and β are defined analogously with respect to D_2. Each of α, $\bar{\alpha}$, β, $\bar{\beta}$ is supposed to consist of finitely many simple arcs, although this condition is lightened in a footnote; $f(t)$ is a piecewise continuous function defined for t on $\alpha+\beta$. The author defines the following functions: (1) $\omega_1(\Delta, z)$ is the harmonic measure of a set Δ on $\alpha+\bar{\alpha}$ at a point z in D_1; (2) $\omega_2(\Delta, z)$ is defined analogously with respect to D_2, β and $\bar{\beta}$; (3) $A(z)=\int_\alpha f(t)d\omega_1(t, z)$ for z in D_1; (4) $B(z)=\int_\beta f(t)d\omega_2(t, z)$ for z in D_2; (5) $\varphi(\Delta, z)=\int_{\bar{\alpha}}\omega_1(\Delta, x)d\omega_1(x, z)$ for Δ on $\beta+\bar{\beta}$ and z in D_1; (6) $a(z)=A(z)+\int_{\bar{\alpha}}B(x)d\omega_1(x, z)$ for z in D_1. In a previous paper [Über das alternierende Verfahren von Schwarz, J. Reine Angew. Math. **180**, 121–128 (1939)], the author proved that the alternating process of Schwarz, as

applied to D_1, D_2 and $f(t)$, converges to the solution $u(z)$ of the Dirichlet problem in D corresponding to the boundary values $f(t)$. It was also shown that $u(z)$ satisfies the integral equation (7) $u(z)=a(z)+\int_\beta u(y)d\varphi(y,z)$ for z in D_1 and a similar equation for z in D_2. In the present paper it is shown that the equation (7) may be solved by the method of successive substitutions. The resolvent kernel $\Phi(\Delta,z)=\sum_{\nu=1}^{\infty}\varphi_\nu(\Delta,z)$, $\varphi_\nu(\Delta,z)=\int_\beta\varphi_{\nu-1}(\Delta,y)d\varphi(y,z)$, $\varphi_1\equiv\varphi$, converges if the set Δ on β is at a positive distance from $\alpha+\beta$, and uniformly in z, for z in any closed set inside D_1. The solution of (7) is given by (8) $u(z)=a(z)+\int_\beta a(y)d\Phi(y,z)$ for z in D_1, and by a similar equation for z in D_2. The integral in (8) is in general an improper one, for Φ does not necessarily converge for a Δ which has limit points on $\alpha+\beta$.

The conditions on the boundaries of D_1 and D_2 imposed by Nevanlinna are considerably weaker than those of Schwarz; moreover, it is shown that under Schwarz's conditions, the kernel $\Phi(\Delta,z)$ converges for every Δ on $\bar\beta$, and the integral in (8) is proper.

The latter part of the paper is devoted to obtaining explicit expressions for the harmonic measures of sets on $\alpha+\beta$ with respect to D, and of sets on $\bar\alpha+\bar\beta$ with respect to $D_1\cdot D_2$. These formulas are obtained in terms of $\omega_1(\Delta,z)$ and $\omega_2(\Delta,z)$. All the results proved for D_1+D_2 are shown to hold in analogous form for $D_1\cdot D_2$. *J. W. Green.*

Monna, A. F. Sur une généralisation du problème de Dirichlet pour ensembles bornés mesurables (B) quelconques. Nederl. Akad. Wetensch., Proc. **42**, 745–752 (1939). [MF 421]

Partant des solutions du problème de Dirichlet pour des ensembles bornés respectivement ouverts ou fermés et correspondant à une fonction continue dans tout l'espace, l'auteur fait une extension aux ensembles boréliens par passage successif transfini aux ensembles boréliens des diverses classes. Extension de diverses notions relatives aux deux cas fondamentaux initiaux: régularité, stabilité, extrémale, représentation de la solution par une intégrale de Stieltjes. *M. Brelot* (Bordeaux).

Brelot, M. Sur l'allure à la frontière des fonctions harmoniques, sousharmoniques ou holomorphes. Bull. Soc. Roy. Sci. Liége **1939**, 468–477. [MF 871]

The author applies the theory of subharmonic functions and of the Dirichlet problem to derive results of a type indicated by the title of the paper. We quote two typical results. Let ω denote an open set in Euclidean space of dimension $n\geqq 2$ and let u be subharmonic in ω. If P is a non-isolated point of the boundary of ω, then $u_1(P)$ will denote $\overline{\lim}\,u(Q)$ for the point Q approaching P from ω, while $u_2(P)$ will denote $\overline{\lim}\,u_1(R)$ for the point $R(\neq P)$ approaching P on the boundary of ω. The author proves that if $u_1(P)<+\infty$ and if P is a regular boundary point, then $u_2(P)=u_1(P)$. For the special case when $n=2$ and u is the absolute value of an analytic function $f(z)$, the author shows that the assumption of the regularity of P can be dropped. *T. Radó.*

Hibbert, Lucien. Propriétés de la fonction harmonique $\log R$ définie sur le cercle-unité par des suites particulières de ses valeurs. C. R. Acad. Sci. Paris **209**, 718–720 (1939). [MF 804]

The author extends and gives a new proof of a theorem of a previous note [the same vol., 287; cf. these Rev. **1**, 56 (1940)]. Let $f(z)=Re^{iV}$ be an analytic function inside the unit circle C; A_1, A_2, D, B are points on C ordered in the negative sense. A perfect set E on C lies between A_1 and A_2, the contiguous intervals of which are l_1, l_2, $\cdots$. Boundary values of $\log R$ are given on C in the following manner: $\log R$ is monotone increasing from $\log R_B$ at A_1 to $\log R_D$ at A_2, being constant on each l_i; from A_2 to D, $\log R=\log R_D$; from D to B, $\log R$ decreases linearly from $\log R_D$ to $\log R_B$; on BA_1, $\log R=\log R_B$. Under these conditions, (1) f' has no zero inside C and exactly one zero on each l_i; (2) $\partial V/\partial n$ is positive on BD, where n is the interior normal; (3) E can be divided into E_L+E_D, where $V=+\infty$ on E_L and V is discontinuous on E_D. For the proof, the author extends V across the intervals l_i and notes that the family of curves $V=$constant is independent of the interval used. From properties of this family the theorem follows. *J. W. Green* (Rochester, N. Y.).

MECHANICS

★**Appleby, M. Elementary Statics.** Cambridge University Press, Cambridge, England, 1939. viii+164 pp. $2.25.

This book deals mainly with equilibrium problems of two dimensional structures in which elastic effects are negligible. Graphical methods and problems involving friction are included. The text is addressed to a student just beginning calculus, and is well adapted to developing technique in solving problems. *P. Franklin* (Cambridge, Mass.).

Arrighi, Gino. Solido con un punto fisso: caso in cui un impulso attivo non cimenta il vincolo. Boll. Un. Mat. Ital. **1**, 339–345 (1939). [MF 577]

If an impulse is applied at a point P of a rigid body free to rotate about a point Q (not the center of gravity), an impulsive reaction will be produced at Q. The locus of points P such that this reaction is zero, for some direction of a non-vanishing impulse, is shown to be either a hyperbolic paraboloid or a pair of planes. The directions of the impulses for the various cases arising when the locus is a pair of planes is studied in detail. *D. C. Lewis, Jr.*

Klotter, K. und Kotowski, G. Über die Stabilität der Bewegungen des Pendels mit oszillierendem Aufhängepunkt. Z. Angew. Math. Mech. **19**, 289–296 (1939). [MF 491]

A physical rigid pendulum is considered whose point of support is moved sinusoidally in a direction making an angle δ' with the vertical. If the momentary deviation of the pendulum is $\psi(t)$, small oscillations are considered of the form $\psi(t)=\alpha+\phi(t)$, where α is a constant and $\phi(t)\ll 1$. The problem then reduces to the following linear inhomogeneous differential equation (of Mathieu type)

$$\frac{d^2\phi}{dx^2}+\phi\left\{\frac{g}{l\omega^2}\cos\alpha+\frac{A}{l}\cos(\delta'-\alpha)\cdot\cos x\right\}=-\frac{g}{l\omega^2}\sin\alpha+\frac{A}{l}\sin(\delta'-\alpha)\cdot\cos x,$$

where $x=\omega t$ and l is the reduced length of the pendulum. It appears that the pendulum can oscillate not only about the mean positions $\alpha=0$ and $\alpha=\pi$ (the stable and unstable positions of equilibrium), but also round positions $\alpha\neq 0$,

$a \neq \pi$. The latter stable positions are designated as "Mittellagen" and are defined by

$$\alpha = \lim_{x \to \infty} \frac{1}{x} \int_0^x \psi(\xi) \cdot d\xi.$$

Solutions are then considered of the form $\phi(x) = \phi_1(x) + \phi_2(x)$, where $\phi_1(x)$ is a solution of the corresponding homogeneous equation and $\phi_2(x)$ is a particular integral, from which the two conditions follow

$$\lim_{x \to \infty} \frac{1}{x} \int_0^x \phi_{1,2}(\xi) \cdot d\xi = 0.$$

The further integration is thereupon effected with the aid of a method due to van der Pol [Physica 5, 157 (1925)], applicable for rapid and small oscillations of the point of support, and where only the central terms of the Hill-determinant could be retained. Curves illustrate the results which confirm earlier investigations of Greenhill, Hirsch and Klotter. A divergence with results obtained by Erdélyi is traced as being due to a different definition of stability used by the latter. *B. van der Pol* (Eindhoven).

Seitter, Hans. Die kleinen Schwingungen eines stark durchhängenden Seiles. Z. Angew. Math. Mech. **19**, 211–215 (1939). [MF 141]

Ein vollkommen biegsames, unausdehnbares Seil ist mit seinen Enden an zwei in der gleichen Horizontalen liegenden Punkten befestigt. Das Seil steht unter dem Einfluss seines Eigengewichtes, welches so verteilt ist, dass die Gleichgewichtsform des Seiles symmetrisch in Bezug auf die Vertikale durch den tiefsten Punkt ist. Es werden die in der Ebene der Gleichgewichtsform stattfindenden kleinen, symmetrischen Schwingungen des Seils um seine Gleichgewichtslage untersucht. Routh hat in seiner Dynamik der Systeme starrer Körper die partielle Differentialgleichung des Problems aufgestellt, in der als unbekannte Funktion die Schwankung des Neigungswinkels der Tangente an die Seilkurve und als unabhängige Veränderliche der Wert dieses Neigungswinkels in der Ruhelage und die Zeit auftreten. Die sich aus der Routh'schen Differentialgleichung ergebende gewöhnliche Differentialgleichung für die Eigenfunktionen wird durch einen Potenzreihenansatz integriert und es wird insbesondere der niedrigste Eigenwert für das homogene Seil bestimmt. *W. Prager* (Istanbul).

Dynamics, Celestial Dynamics, Relativity

Lewis, Daniel C., Jr. Contributions to the transformation theory of dynamics. Trans. Amer. Math. Soc. **46**, 374–388 (1939). [MF 470]

The author defines an (analytic) transformation from x_i $(i=1, \cdots, n)$ to $\bar{x}_i$ $(i=1, \cdots, n)$ as Pfaffian with respect to a linear differential form $\sum X_i(x)dx_i$ if the difference $\sum(X_i(x)dx_i - X_i(\bar{x})d\bar{x}_i)$ is an exact differential in the region R of $(x_1 \cdots x_n)$-space under consideration. The discrete groups formed by such a Pfaffian transformation T and the iterates of T are intimately analogous to continuous groups of contact transformations defined by a set of differential equations of dynamical type; and Lewis's paper is best understood in the light of this analogy. Certain of the results concerning such a transformation had been stated independently by the reviewer [Mem. Pontif. Acad. Sci. Novi Lyncaei (3) 1, 1935], but proofs are here given for the first time; others are generalizations of results established in the much simpler case $n=2$ by the reviewer.

The Pfaffian transformation T under consideration is reduced at the outset to the nonsingular case $n=2k$ when the skew symmetric determinant of

$$a_{ij} = \left| \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right|$$

is of rank $2k$. He shows that if there exists an invariant subspace of R which is invariant under T, then the transformation is also Pfaffian in the subspace, a result to be compared with well-known facts concerning invariant subspaces for continuous one-parameter groups of contact transformations. He proves also that the characteristic multipliers occur in reciprocal pairs, also in clear analogy with the continuous case. Lewis then proves further that with T (or at least a suitable iterate thereof) there is associated a set of formal differential equations

$$\sum_i a_{ij} \frac{dx_j}{dt} = \frac{\partial Q}{\partial x_i},$$

where Q is a formally invariant power series in $x_1, \cdots, x_n$ without linear terms, such that the formal series for $x_i(t)$ which reduce to $x_1, \cdots, x_n$ for $t=0$ become $\bar{x}_1, \cdots, \bar{x}_n$, respectively, for $t=1$. He shows that by a suitable choice of variables these formal differential equations may always be given Hamiltonian form. *G. D. Birkhoff.*

Lewis, Daniel C., Jr. Formal power series transformations. Duke Math. J. **5**, 794–805 (1939). [MF 820]

Let T be an analytic nonsingular transformation of the neighborhood of the origin in $(x_1 \cdots x_n)$-space into itself. Then all the iterates of T, $T^{\pm 1}$, $T^{\pm 2}$, $\cdots$ are of similar type. Lewis shows how it is always possible to construct (formal) power series in the variables $x_1, \cdots, x_n$, with coefficients simple functions of t (polynomials in t and a finite number of expressions of the form e^{at}), such that for all integral values of m, T^m is obtained from these series in $x_1, \cdots, x_n$ by setting $t=m$. The case when T is linear is especially studied.

It is furthermore shown that, in the "pseudo-incommensurable case," to which some T^m is proved always to belong, there is an associated set of formal differential equations,

$$\frac{dx_i}{dt} = U_i,$$

where the series $U_i(x)$ are without constant terms. This result is applied in the paper reviewed above to the dynamical case. *G. D. Birkhoff* (Cambridge, Mass.).

Morse, Marston and Hedlund, Gustav A. Symbolic dynamics II. Sturmian trajectories. Amer. J. Math. **62**, 1–42 (1940). [MF 952]

The authors consider cell sequences, which are essentially ordered sets of nonnegative integers $\cdots B_{-1}B_0B_1 \cdots$; a cell sequence may be finite (cell chain), or it may extend to infinity in either direction (cell beam), or it may extend to infinity in both directions (cell series). The b-length of a chain is the sum of the numbers which make up the chain. A cell sequence is Sturmian if the b-lengths of any two chains made up of the same number of consecutive elements of the sequence differ by at most one. If b_n is the b-length of an arbitrary chain made of n consecutive elements of a Sturmian beam or series, then b_n/n tends to a finite limit α,

the frequency of the sequence, as n becomes infinite. If α is a rational number q/p (with p and q relatively prime), a series with frequency α is either periodic with minimum period p or else it is composed of two periodic beams with period p, one to the right and one to the left, which have a chain of $p-2$ elements in common; these last are called skew series. A periodic series is characterized by its frequency; a skew series is characterized by its frequency and by the b-length of the chain in which its component beams intersect plus one element on each end of this chain.

To each Sturmian series there corresponds in a trivial fashion a trajectory, as defined by the authors in an earlier paper [Amer. J. Math. 60, 815–866]. If the frequency of a series is irrational, the corresponding trajectory is not periodic but it is recurrent (for each integer m there exists an integer n such that every block of n consecutive elements of the trajectory contains a copy of every block of m consecutive elements which appears anywhere in the trajectory). This leads to a new set of nonperiodic recurrent trajectories whose recurrency functions are computed; from this computation the authors are able to deduce several facts concerning the asymptotic behavior of the functions. There is a final section showing how Sturmian series may be used to characterize the distribution of the zeros of the solutions of some second order differential equations.

C. B. Tompkins (Princeton, N. J.).

Wintner, Aurel. On the almost periodic behavior of the lunar node. Amer. J. Math. 62, 49–60 (1940). [MF 954]

Let $z''+f(t)z$ be the equation of variation of the third coordinate z of a planar solution of the restricted problem of three bodies. The author reduces this equation by a linear substitution (using u and v in terms of z and z') to a system of two linear differential equations in such a way that the node $\theta(t)$ of the approximate non-planar solution of three bodies is identical with the polar angle of the corresponding path in the (u, v)-plane. Thus not only does this node have a mean motion ω, but the remainder term $\theta(t)-\omega t$ is uniformly almost periodic. *E. R. van Kampen.*

Sterne, Theodore Eugene. The gravitational motion of a pair of rigid bodies. Proc. Nat. Acad. Sci. U. S. A. 25, 468–474 (1939). [MF 878]

The author treats the problem by the method of small oscillations about steady motion. Only librations in the orbital plane are discussed. It is shown that, depending upon the physical constants for the system, the motion may be stable or unstable, and, if the motion is stable, the line of apsides may advance or regress. These results conform to those derived earlier by a different method by Walter [Schr. Königsberg. Gel. Ges. 10, 4 (1933)]. The problem was originally suggested by a study of the motion of eclipsing binary stars; it is now generally agreed that the rigid-body solution does not apply to gaseous (or liquid) stars. The unstable solutions, of no immediate astronomical interest, are not considered in detail. They represent cases in which one or both components make complete rotations relative to the state of steady motion. A different form of solution would be required. *D. Brouwer.*

Buchanan, Daniel. Asymptotic isosceles-triangle solutions for unequal masses. Rend. Circ. Mat. Palermo 62, 1–10 (1939). [MF 713]

If two finite bodies of equal mass, subject to no forces except their mutual gravitational attraction, move in circles about their common center of gravity, then it is possible to project an infinitesimal body from the center of gravity along the normal to the plane of their motion so that its motion will be periodic. It is obvious that the orbit of the infinitesimal body will be a straight line. The problem considered in this paper is to determine orbits which will approach the linear orbit asymptotically as the finite bodies become unequal. The differential equations of motion are referred to a set of axes rotating so that the coordinates of the finite bodies are constants. Series expansions for the solutions are found in the form:

$$x=\sum x_j\epsilon^j, \quad y=\sum y_j\epsilon^j, \quad z=\psi(t)+\sum z_j\epsilon^j,$$

where ϵ is half the difference of the finite masses, $\psi(t)$ is periodic and each x_j, y_j and z_j is of the form $e^{\lambda t}P(t)$, $P(t)$ being periodic. If the real part of λ is different from zero and negative, then the solutions obtained will approach the generating solution $x=y=0$, $z=\psi(t)$ as t approaches plus infinity. *H. E. Buchanan* (New Orleans, La.).

MacColl, L. A. Geometric aspects of relativistic dynamics. Trans. Amer. Math. Soc. 46, 328–347 (1939). [MF 467]

The author determines a set of geometrical properties characteristic of the 3-parameter families of trajectories of a relativistic particle moving in a plane under general forces which are functions of position only. The differential equations of the motion are

$$\frac{d}{dt}(\beta\dot{x})=\phi(x, y), \quad \frac{d}{dt}(\beta\dot{y})=\psi(x, y),$$

where $\beta=\{1-(\dot{x}^2+\dot{y}^2)/c^2\}^{-\frac{1}{2}}$, and the third-order differential equation of the trajectories is obtained by eliminating t. An examination of this differential equation shows that there are six properties characteristic of the families of trajectories as against five in the classical case discussed by Kasner [Trans. Amer. Math. Soc. 7, 401–424 (1906) and Differential-Geometric Aspects of Dynamics, Amer. Math. Soc. Colloquium Publ., vol. 3_2, 9–17 (1913)]. Four of the properties in the relativistic case correspond to four of the classical properties, while the remaining two have no classical analogue. The author also brings to light another difference between the relativistic and classical cases: in the former but not in the latter the geometry of the family of trajectories for a particle of given proper mass determines the field of force completely. He further shows that the most general point transformation which converts every family of plane dynamical trajectories into a family of plane dynamical trajectories consists of rigid motions, magnifications and reflections in a straight line. He concludes by considering the 3-dimensional motion of a particle in the "natural" case when the total energy is prescribed and the field of force is derived from a potential energy, and finds that the characteristic properties of the family of trajectories is the same as those found by Kasner for the classical case. *H. S. Ruse* (Southampton).

Lämmel, R. Masse und Energie in der speziellen Relativitätstheorie. Helvetica Phys. Acta 12, 511–518 (1939). [MF 596]

Schiff, L. I. A question in general relativity. Proc. Nat. Acad. Sci. U. S. A. 25, 391–395 (1939). [MF 887]

Two concentric spheres carry equal and opposite electric

charges. If the spheres are fixed in a Galilean frame they produce no external field, but if they rotate about a common diameter they produce a magnetic field. The paper is concerned with the question: What field is observed by an observer who rotates with the spheres? The author claims that his argument shows that no field is observed.

J. L. Synge (Toronto, Ont.).

Chou, P. Y. On the method of finding isotropic static solutions of Einstein's field equations of gravitation. Amer. J. Math. **62**, 43–48 (1940). [MF 953]

Continuing an earlier paper [Amer. J. Math. **59**, 754 (1937)], the author seeks line-elements in empty regions of space-time of the form

$$U^2dt^2-f^4(dx^2+dy^2+dz^2),$$

where U and f are functions of x, y, z. The field equations (with no cosmological term) impose the condition that f must be harmonic, but the completion of the general three-dimensional problem is not attempted. By taking

$$f=F(x+iy)+F(x-iy),$$

the author solves the two-dimensional problem, obtaining as the only solutions Kasner's solution [Trans. Amer. Math. Soc. **27**, 160 (1925)] and the field of the semi-infinite plane, as given in his earlier paper. *J. L. Synge.*

Moghe, D. N. On isotropic manifolds in the theory of relativity. Proc. Indian Acad. Sci., Sect. A. **10**, 275–278 (1939). [MF 688]

A discussion of the connection between the isotropy conditions $T_1^1=T_2^2=T_3^3=-p$ imposed upon the energy-momentum-stress tensor, and the resulting differential-geometric isotropy of the spaces $t=$const.

H. P. Robertson (Princeton, N. J.).

Moghe, D. N. On some non-static solutions of Einstein's gravitational equations, and fluid spheres with the pressure and density as slowly varying functions of time. Proc. Indian Acad. Sci., Sect. A. **10**, 407–416 (1939). [MF 914]

The author attempts to discuss solutions of the equations for non-static fluid spheres by imposing ad hoc assumptions on the form of the line element. He (incorrectly) assumes that for a perfect fluid the spatial diagonal elements of the stress tensor must be equal. The discussion of the special solutions is for the most part inadequate or incorrect: thus Case I does not satisfy his own criterion of isotropy; Case II is needlessly restricted, and does not admit the imposition on an equation of state of the form $\rho=\rho(p)$; and Case III is invalidated by an arithmetical error in the derivation of equation (18). Although the author announced his intention at the outset to offer non-static solutions which can represent a state of stable equilibrium, no investigation of the stability of his proposed solutions is given.

H. P. Robertson (Princeton, N. J.).

Schrödinger, Erwin. The proper vibrations of the expanding universe. Physica **6**, 899–912 (1939). [MF 752]

The second-order wave equation is generalized to curved space by merely writing down the corresponding tensor equation. The equation is solved for the curved spaces of the expanding universes of general relativity, the solutions found being valid in terms of the usual co-moving coordinates. No proof is given that the solutions are valid in other coordinate systems. It appears from these solutions that two proper vibrations with frequencies of opposite sign cannot be rigorously separated. Physically this would mean that, for particles, there would be production or annihilation of matter as the universe expanded. It is shown that this result can be avoided if the expansion proceeds linearly with the time. In general, it is shown that a wave function shares the general dilatation so that the wave-lengths of material particles and of light waves increase proportionally to the radius of curvature of space. But if de Broglie's formula for material particles is assumed to be valid in curved space, then different frequencies must be affected by different expansion factors. *G. C. McVittie* (Eastbourne).

Whitrow, G. J. On the Lobatchewskian trigonometry of a static substratum. Quart. J. Math., Oxford Ser. **10**, 313–319 (1939). [MF 1041]

The paper deals with Milne's cosmological result relating to families of time scales t and τ by the relation $\tau=t_0\log(t/t_0)+t_0$ [Proc. Roy. Soc. London, Ser. A. **158**, 324 (1937)]. It is shown that for observers using the t scale the trigonometry of rapidity vectors (defined by $\tilde{\omega}=(\tanh(V/c))^{-1}$; V relative velocity) is that of a hypersphere of radius i. From this Milne's result can be deduced; for observers using the τ scale there exists a position space, hyperbolic and of constant negative curvature $-(ct_0)^{-2}$.

L. Infeld (Toronto, Ont.).

NUMERICAL AND GRAPHICAL METHODS

★**Voellmy, Erwin. Fünfstellige Logarithmen und Zahlentafeln.** Orell Füssli, Zürich, 1939. 184 pp. RM 2.70.

Moon, Parry. A table of Fresnel reflection. J. Math. Phys. Mass. Inst. Tech. **19**, 1–33 (1940). [MF 873]

These tables give values of

$$\left\{\frac{\cos\phi-(m^2-\sin^2\phi)^{\frac{1}{2}}}{\cos\phi+(m^2-\sin^2\phi)^{\frac{1}{2}}}\right\}^2 \quad\text{and}\quad \left\{\frac{m^2\cos\phi-(m^2-\sin^2\phi)^{\frac{1}{2}}}{m^2\cos\phi+(m^2-\sin^2\phi)^{\frac{1}{2}}}\right\}^2,$$

together with the first differences for $0\leqq\phi\leqq\pi/2$ in steps of 0.02 radians (for $1.5<\phi<\pi/2$ even in steps of 0.005) for $m=1.1, 1.2, \cdots, 1.9$ and $4/3$. The numbers are given to 8 figures (computation having been carried out to 10 figures).

W. Feller (Providence, R. I.).

Lowan, Arnold N. and Laderman, Jack. On the distribution of errors in nth tabular differences. Ann. Math. Statistics **10**, 360–364 (1939). [MF 756]

Let $f(x)=1/(2a)$ for $|x|<a$ and $f(x)=0$ for $|x|>a$, and let η_i, $i=0, \cdots, n$, be independent random variables with the probability density $f(x)$. The authors calculate, using characteristic functions, the probability distribution of

$$\eta=\binom{n}{0}\eta_0-\binom{n}{1}\eta_1+\cdots+(-1)^n\binom{n}{n}\eta_n.$$

Explicit formulas are given for $n=1, 2, 3$. If η_i are the errors of table entries x_i, due to rounding off, then η is the error of $\Delta^n x_0$. *W. Feller* (Providence, R. I.).

Camp, Kingsland. Actuarial note: practical interpolation methods with second-order curves. Actuar. Soc. America Trans. **40**, 426–439 (1939). [MF 530]

Stock, J. Stevens. A method of graphic interpolation. J. Amer. Statist. Assoc. **34**, 709–713 (1939). [MF 701]

The author calls attention to Aitken's geometrical interpretation of his iteration method for interpolation [Proc. Edinburgh Math. Soc. (2) **3** (1932)] and gives an extension to the bivariate case. *W. Feller* (Providence, R. I.).

Ballantine, J. P. The best (?) formula for computing π to a thousand places. Amer. Math. Monthly **46**, 499–501 (1939). [MF 362]

Stoyanoff, A. Quelques remarques sur "de circuli magnitudine inventa" de Huygens. Ann. Univ. Sofia. II. Fac. Phys. Math. Livre 1. **35**, 157–166 (1939). (Bulgarian. French summary) [MF 1375]

Computation of π with five decimals based on the inequality $F(u)-R_1(u)\leqq u^4(1+5u)/60$, $u=\sin^2 x/2$, $x\leqq 45°$, where

$$F(z^2)=1+\frac{2}{3}z^2+\frac{2\cdot 4}{3\cdot 5}z^4+\cdots=z^{-1}(1-z^2)^{-1}\arcsin z$$

and

$$R_1(u)=\left(1-\frac{4}{21}u-\frac{4}{105}u^2\right)\Big/\left(1-\frac{6}{7}u\right).$$

J. D. Tamarkin (Providence, R. I.).

Lorey, Wilhelm. Über ein Eulersches Verfahren zur Wurzelberechnung. Monatsh. Math. Phys. **48**, 190–197 (1939). [MF 639]

Discussion of a paper by Euler, 1770 [ser. 1, vol. 6, pp. 240–262, of Opera Omnia]. A systematic method of computing roots of real numbers, applied in detail to indices 2, 3, 4. To find $r^{1/3}$, $r>1$, Euler starts from three recurrence relations $a_n=a_{n-1}+b_{n-1}+c_{n-1}$, $b_n=ra_{n-1}+b_{n-1}+c_{n-1}$, $c_n=ra_{n-1}+rb_{n-1}+c_{n-1}$, a_0, b_0, c_0 essentially arbitrary. Then $\lim (b_n/a_n)=r^{1/3}$, $\lim (c_n/a_n)=r^{2/3}$. The a_n, b_n, c_n satisfy separately the linear difference equation of the third order

$$y_n-3y_{n-1}-3(r-1)y_{n-2}-(r-1)^2y_{n-3}=0.$$

Numerical examples are carried out, and indicate that the method is practical. For $n=3$, the procedure is apparently related to a method making use of Jacobi's "kettenbruchähnlicher" algorithm. [Compare Vahlen, Konstruktionen und Approximationen, 1911, 283–4.]

C. A. Hutchinson informs me that the last digits are frequently incorrect, and points out numerous misprints. *A. J. Kempner* (Boulder, Colo.).

Casale, F. Su di una equazione collegata a quella di Keplero. I, II. Ist. Lombardo, Rend. **72**, 333–346, 347–361 (1939). [MF 930, 931]

The root of the equation

$$e^r(1-r)+e^{-r}(1+r)=0$$

is calculated to 22 decimal places by Newton's method. From this is calculated to 21 places the value of $2re^r/(e^{2r}+1)$, which is the radius of convergence of the Lagrange series for the root of Kepler's equation. *P. W. Ketchum.*

Lehmann, A. Über die Inversion des Gaussschen Wahrscheinlichkeits-Integrals. Mitt. Verein. Schweiz. Versich.-Math. **38**, 15–52 (1939). [MF 304]

A survey of different methods of computing the coefficients of the Taylor series for the inverse function of $\int_0^z e^{-x^2}dx$. *W. Feller* (Providence, R. I.).

Bickley, W. G. Formulae for numerical integration. Math. Gaz. **23**, 352–359 (1939). [MF 355]

Turton, F. J. The errors in the numerical solution of differential equations. Philos. Mag. **28**, 359–363 (1939). [MF 1119]

The errors arising in the step-by-step process of integrating a differential equation numerically may be classified as (1) due to uncertainty of initial values; (2) due to intrinsic errors in the formulae used in the step-by-step method; (3) rounding off errors in (2); (4) random errors. The author makes detailed analysis of these sources of errors and their accumulation as the integration proceeds and concludes: "In the absence of any reliable method of forecasting or finally estimating the error, the only alternative is to endeavor to have no error to the desired number of significant figures. This requires that at least two formulae be used, in which the intrinsic errors are substantially different, to check each other. . . . It will be desirable to carry an extra figure at least to indicate the approach of a discrepancy in the values at the required place, when it will be necessary to reduce the interval." *W. E. Milne.*

Turton, F. J. Two notes on the numerical solution of differential equations. Philos. Mag. **28**, 381–384 (1939). [MF 1121]

(A) The author presents several interpolation formulae useful in the step-by-step numerical solution of differential equations, and especially convenient when it becomes necessary to shorten the interval h. One such formula uses the four values y_0, y_1, y_0', y_1' and has an error of the order of $h^4y^{(4)}$, another uses y_0, y_1, y_{-1}, y_0', y_1', y_{-1}' and has an error of the order of $h^6y^{(6)}$. Still others use 3 values of y and 2 values of y' with an error of the order of $h^5y^{(5)}$, or 2 values of y and 3 of y' with a similar error. A typical form is

$$y_r=y_0(1-r^2)^2+[y_1r^2(1+r)^2+y_{-1}r^2(1-r)(7+5r)]/4 \\ +hy_0'r(1-r)(1+r)^2+[hy'_{-1}r^2(1-r^2)]/2,$$

with an error $-r^2(1+r)^2(1-r)h^5y^{(5)}/5!$. When halving the interval a convenient form is

$$y_{1/2}=(y_{-1}+9y_0+6y_1)/16+h(16y_0'-5y_1'+y'_{-1})/64,$$

with an error $h^5y^{(5)}/3840$. These are suitable in connection with Milne's method of integration, being of the same order of accuracy.

(B) Most step-by-step methods of integration require several consecutive values of y for starting. The author presents several sets of formulae of different orders of accuracy for finding these values. A typical set is

$$y_2=y_0+[h(y_0'+4y_1'+y_2')]/3-h^5y^{(5)}/90,$$
$$y_3=y_1+[h(y_1'+4y_2'+y_3')]/3-h^5y^{(5)}/90,$$
$$y_1=(8y_0+y_3)/9+[h(y_0'+2y_1'-y_2')]/3-h^5y^{(5)}/45,$$

to determine y_2, y_3, y_1 by successive approximations starting with trial values, with corrected values as available, in the order given. *W. E. Milne* (Corvallis, Ore.).

Crout, Prescott D. An application of polynomial approximation to the solution of integral equations arising in physical problems. J. Math. Phys. Mass. Inst. Tech. **19**, 34–92 (1940). [MF 874]

The author states in his introduction: "It seems that the integral equation should be a powerful tool at least comparable in importance with the differential equation, and one in common every day use in solving engineering problems. Such would be the case if there were any way to obtain within a reasonable time solutions having engineering accuracy."

The author gives a very detailed exposition of a method of approximate solution of Fredholm equations based on interpolation by polynomials. The integral equation is reduced to a system of $2n+1$ linear equations by writing the unknown function as a linear combination $\sum a_i P_i(x)$ of certain standard polynomials and then requiring the integral equation to be satisfied at $2n+1$ equally spaced points in the range of integration. The standard polynomial $P_i(x)$ is taken to be the polynomial of degree $2n+1$ which equals unity at the ith of these points and vanishes at the other points. By a linear transformation, these points may be sent into the particular points $-n, \cdots, -1, 0, 1, \cdots, n$. It is thus possible to tabulate values for these transformed polynomials $K_i(\lambda)$, which the author does for λ by tenths and for $n=1, 2, 3$. Tables of $K_i'(\lambda)$, $K_i''(\lambda)$ and $K_i'''(\lambda)$ for integral values of λ and the same values of n are also included.

The author illustrates the method by solving three well-known problems: (a) The buckling of a slender column with two hinged ends; also with one end fixed and one hinged. (b) The stress distribution in the area of contact of two rollers. (c) The distribution of induced charge on a long, uncharged, perfectly conducting cylinder in a uniform field. The requisite integral equations are derived and the practical aspects of carrying out the solution discussed in greatest detail. Using $n=2$ the results are within 2% of the known exact solutions. A considerable amount of intuitive evidence is presented for the validity of the method, but there is no rigorous consideration of the set of equations for which the method will work. There is also a comparison with the methods of Ritz and Raleigh.

P. W. Ketchum (Urbana, Ill.).

Pipes, Louis A. The method of symmetrical components applied to harmonic analysis. Philos. Mag. **29**, 66–74 (1940). [MF 1159]

The author discusses some practical details of the interpolation to $2r+1$ equally spaced values by means of an exponential polynomial of the form $\sum_{-r}' A_n e^{in\theta}$.

P. W. Ketchum (Urbana, Ill.).

Debye, P. Untersuchung eines neuen Vorschlags zur Fourier-Analyse von Elektronenaufnahmen. Phys. Z. **40**, 573–577 (1939). [MF 400]

Concerning an earlier paper of Pauling and Brockway [J. Amer. Chem. Soc. **57**, 2684 (1935)]. See also criticisms by Debye and Pirenne [Ann. Physik (5) **33**, 617 (1938)].

Harkink, F. Die Brunsviga-Koordinatenmaschine. Allg. Vermessgs-Nachr. **51**, 597–602, 613–618 (1939). [MF 906]

Jelstrup, Gunnar. Mathematical instruments. Norsk Mat. Tidsskr. **21**, 81–98 (1939). (Norwegian) [MF 696]

Lecture giving a general survey of the theory of the simplest instruments. *W. Feller* (Providence, R. I.).

Rosseland, Svein. Mechanische Integration von Differentialgleichungen. Naturwissenschaften **27**, 729–735 (1939). [MF 740]

The author describes the differential analyzer at present in operation in the Institute of Astrophysics, Blindern, near Oslo, Norway. This machine for the integration of ordinary differential equations has 12 integrators and 7 input tables. Bearing in mind that the "capacity" of the analyzer, that is, the highest order of the differential equation which can be solved with it, depends essentially on the number of integrators available, the author points out that the Oslo machine has the greatest capacity of any one built up to the present. (So far as known to the reviewer, there are at present 7 differential analyzers built or almost completed, as follows: (1) Original machine at Mass. Inst. Tech., Cambridge, Mass., with 6 integrators, built in 1930 by V. Bush; (2) Univ. of Penn., Philadelphia, with 10; (3) Aberdeen Proving Ground, Aberdeen, Md., with 10; (4) Univ. of Manchester, England, with 8; (5) Inst. Astrophys., Blindern, Norway, with 12; (6) Leningrad, U.S.S.R., with 6 (?); (7) New machine at Mass. Inst. Tech., with 18. No details seem to have been published about the Leningrad machine.)

The construction of the Oslo machine, described in the paper, is closely along the lines of the first Bush machine, the former differing only by refinements designed to reduce the number of gear connections and the number of shaft bearings, so as to minimize all frictional losses. No quantitative information on the overall precision of the machine is given.

Simple examples of equations solved by means of the Oslo machine are: $y''=-xy$, $y''+y^2=2ky$, $\varphi''=-\sin\varphi$, $x^2y''+y^3=0$, and $u''+u=f/u^2h^2$. Solutions for different boundary conditions are exhibited. The paper closes with remarks on the kind of problems which can be solved by the differential analyzer, such as the motion of a charged particle in the field of a magnetic dipole [cf. G. Lemaitre and M. S. Vallarta, Phys. Rev. **49**, 719 (1936)] and points out that the restricted three body problem, which requires 10 integrators, is comfortably within reach of the Oslo machine.

The cost of building another machine similar to that at Oslo is estimated at 70,000 German marks. (This seems moderate compared with the cost of the two machines now at Mass. Inst. Tech.) *M. S. Vallarta.*

Brown, S. Leroy. A mechanical harmonic synthesizer-analyzer. J. Franklin Inst. **228**, 675–694 (1939). [MF 721]

A harmonic synthesizer having fifteen sine and fifteen cosine components is described. The machine closely resembles the Kelvin tide predictor in construction, and is capable of accurately drawing resultant curves for ratios of

harmonic amplitudes as great as 200. Use of the synthesizer for analysis by the method of Kranz [J. Franklin Inst. **204**, 245–262] is illustrated, and the method extended to the determination of high order harmonics. *R. L. Dietzold.*

Blackett, P. M. S. and Williams, F. C. An automatic curve follower for use with the differential analyser. Proc. Cambridge Philos. Soc. **35**, 494–505 (1939). [MF 550]

The slope of a curve to be followed is measured by a photocell system, and a mechanical integrator then reproduces the original curve by integration of the slope. The method of measuring slope is similar to that described by Sears [J. Opt. Soc. Amer. **25**, 162–164 (1935)]. A method is shown for causing the device to find the curve if departures become momentarily large. *S. H. Caldwell.*

Williams, F. C. A reversible head for the automatic curve following device. Proc. Cambridge Philos. Soc. **35**, 506–511 (1939). [MF 551]

In a companion paper [see above review] Blackett and Williams describe a curve follower which operates in only one direction of motion along the axis of abscissae. The present paper describes a modification to permit operation in both directions. *S. H. Caldwell* (Cambridge, Mass.).

Lennard-Jones, J. E., Wilkes, M. V. and Bratt, J. B. The design of a small differential analyser. Proc. Cambridge Philos. Soc. **35**, 485–493 (1939). [MF 549]

Design details are given for the construction of an inexpensive differential analyzer, using many standard parts. The machine is essentially of the type described by Hartree and Porter [Proc. Manchester Lit. Phil. Soc. **79**, 51 (1935)]. *S. H. Caldwell* (Cambridge, Mass.).

Fourcade, H. G. A projection method of mapping from air photographs. Trans. Roy. Soc. South Africa **27**, 321–367 (1940). (1 plate) [MF 1084]

A new precision instrument, called a "stereoprojector," is described which makes it possible to construct maps from the coincidences of projected images of two aerial photographs. "The object of the new method detailed in the present paper is to provide means whereby both images, which may have been photographed at different heights, remain in sharp focus at all projection-levels, while their geometrical relations remain mathematically accurate, and, at the same time, a wide range of scales for the constructed map is made available." The author, whose "stereogoniometer" for the determination of the relative tilts in space of pairs of aerial photographs is widely used, gives the theory of his new apparatus and a detailed description of the model which has been actually built.

The stereoprojector uses pictures which are rectified to a horizontal plane. This rectification can be made with a rectifying camera to work in conjunction with the stereoprojector. The necessary determination of tilts can be made in a stereogoniometer or, using a formula developed in this paper, in the stereoprojector itself. According to the author, the instrument works very satisfactorily.

E. Helly (Paterson, N. J.).

Rinner, Karl. Beitrag zur Orientierungsaufgabe. Allg. Vermessgs-Nachr. **51**, 520–525 (1939). [MF 907]

A new and rather simple derivation of an equation for the correspondence setting of two aerial photographs is given together with the resulting equations for the errors. *E. Helly* (Paterson, N. J.).

Rinner, Karl. Eine elementare Ableitung der Fehlergleichungen für den Normalfall der gegenseitigen Orientierung. Allg. Vermessgs-Nachr. **51**, 661–665 (1939). [MF 1400]

The influence of errors of correspondence setting (orientation) of two aerial photographs on the ground position of points is computed, in an elementary way, by adding the effects of errors due to small rotations around the three axes of swing, setting and tilt. A few diagrams are added, representing the deformation of the model due to the errors of orientation. *E. Helly* (Paterson, N. J.).

Feyer, Edwin. Über Fehlerwirkung bei der Ermittlung des Uhrstandes durch Aufnahme von Koinzidenzzeitsignalen. Allg. Vermessgs-Nachr. **52**, 27–33 (1940). [MF 1304]

BIBLIOGRAPHICAL NOTES

***Comptes Rendus du Congrès des Sciences Mathématiques de Liége. Exposition Internationale de l'Eau, Liége, 1939.** G. Thone, Liége, 1940. 150 pp.

This volume contains the following lectures: Paul Montel: La géométrie des polynomes; J.-G. van der Corput: Sur la théorie additive des nombres; Elie Cartan: Sur quelques familles remarquables d'hypersurfaces; J.-A. Barrau: La cinématique dans le groupe des similitudes du plan; S. Mandelbrojt: Les fonctions indéfiniment dérivables; Henri Lebesgue: Les n-sectrices d'un triangle: extension d'un théorème de Frank-Morley; W. Van der Woude: Sur l'application du "théorème fondamental de l'algèbre" de Noether; Otto Blumenthal: La géométrie des polynomes binomiaux.

In addition, there are communications on different topics by Pol Burniat, L. Derwidué, A. Errera, L. Gautier, R.-H.-J. Germay, Paul Gillis, Lucien Godeaux, Guy Hirsch, M. Kraitchik, T. Lemoyne, Th.-H. Lepage, P. Libois and P. Defrise, M. Linsman, H. Lorent, O. Rozet, Fernand Simonart, J. Teghem, and G. Van der Lijn.

***Selecta. Jubilé scientifique de M. Émile Borel.** Gauthier-Villars, Paris, 1940. 418 pp.

Contains 27 reprints of Borel's publications, together with some short commentaries by Denjoy, Fréchet, Lévy and Valiron, and also a full bibliography of Borel's work.

Indagationes Mathematicae

Beginning with 1939, the Koninklijke Nederlandsche Akademie van Wetenschappen is publishing a new mathematical journal Indagationes Mathematicae, which appears five times a year in January, March, May, September and November. All mathematical papers appearing in the publications of the academy are collected in this journal, and no others.

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